ClassRoom

How to Prove It

In this episode of "How To Prove It", we prove a simple yet surprising collinearity associated with any triangle. We do so in two different ways and then contrast the two proofs. It would be instructive for a student to study these proofs and also the remarks made at the end.

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In $\triangle ABC$, let the feet of the altitudes from A, B, C be D, E, F respectively. Select any one altitude, say AD, and from its foot (D), drop perpendiculars to the other two sides (AB, AC) and to the other two altitudes (BE, CF). Let the feet of these perpendiculars be P, S and Q, R, as shown in Figure 1. Then the points P, Q, R, S lie in a straight line.

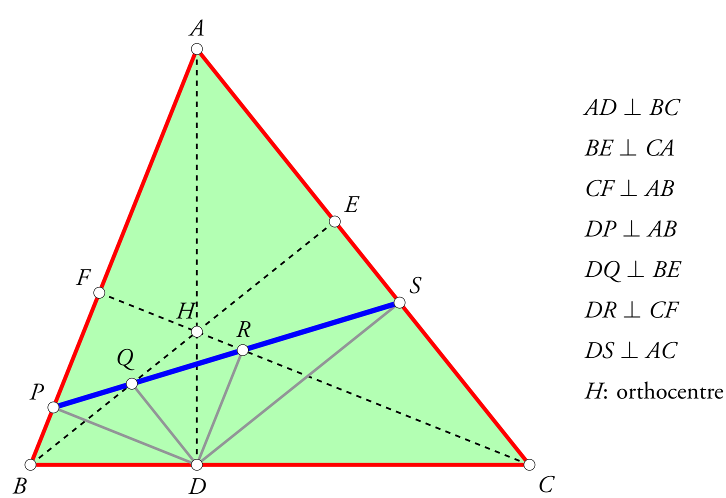


Figure 1

Keywords: *Altitudes, collinearity, cyclic quadrilateral, angle chasing, supplementary, coordinate, slope, linear equation, pure geometry, proof*

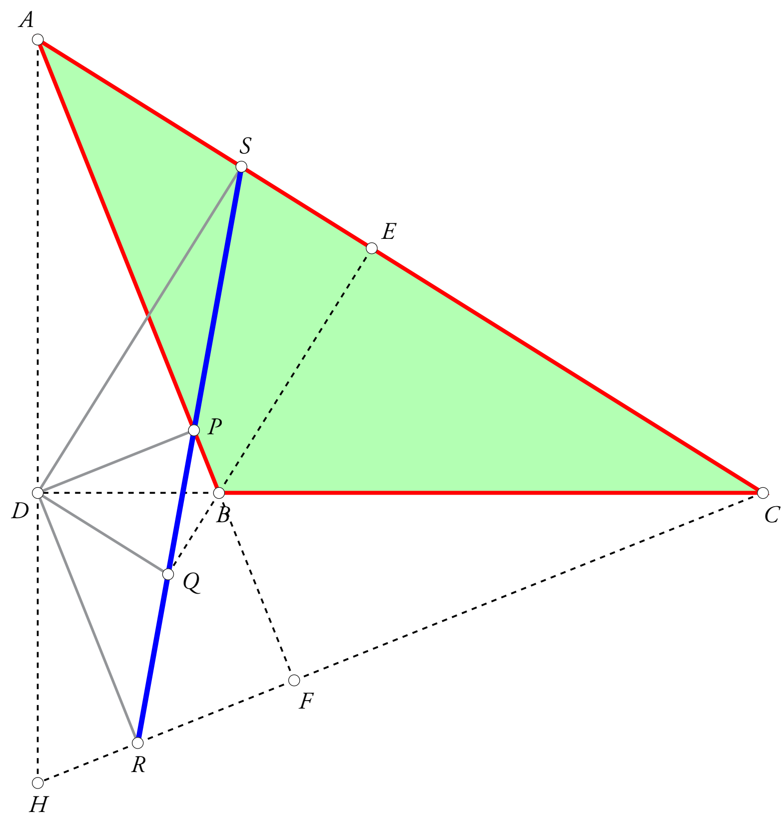


Figure 2

In case you are wondering how the configuration will change if the triangle is obtuse, please study Figure 2 (in which $\angle B$ is obtuse). Many of the points now fall outside the given triangle. The conclusion however stays intact.

We offer two proofs of the proposition. The reader is invited to judge the relative merits of each proof.

First proof

This proof uses nothing more than 'angle chasing'. We start by redrawing the figure, taking care to use different colours for segments PQ, QR and RS (see Figure 3). The purpose of doing this is simply to avoid the error of implicitly assuming the very thing you are trying to prove—an error that is particularly easy to make in geometry.

Note the presence of a number of cyclic quadrilaterals; this is natural, given the number of perpendiculars that have been drawn. Note also that we have coloured some of the quadrilaterals differently.

To prove that points P, Q, R lie in a straight line, it suffices to show that $\angle PQD$ and $\angle RQD$ are supplementary. We show this as follows. Since quadrilateral $PBDQ$ is cyclic, it follows that $\angle PQD = 180^\circ - \angle B$. Since quadrilateral $QDRH$ is cyclic, it follows that $\angle RQD = \angle RHD$. But $\angle RHD = \angle FBD$ (this follows because quadrilateral $FBDH$ is cyclic), i.e., $\angle RHD = \angle B$. It follows that $\angle PQD$ and $\angle RQD$ are supplementary. Hence points P, Q, R lie in a straight line.

In exactly the same way we show that points Q, R, S lie in a straight line. It follows that all four points P, Q, R, S lie in a straight line, as required. $\square$

Second proof

This time we shall use coordinates. Adopt a system of coordinates in which line BC represents the x-axis, and line AD represents the y-axis; so D serves as the origin. Let the coordinates of the vertices be as follows: $A = (0, a)$, $B = (b, 0)$, $C = (c, 0)$. Note that a, b, c do not represent the lengths of the sides of the triangle. We now compute the coordinates of the points P, Q, R, S in terms of a, b, c.

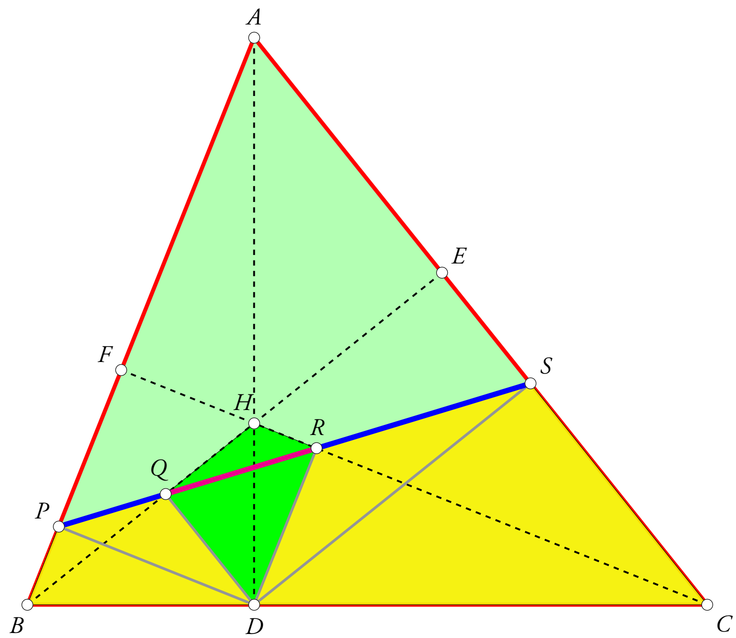


Figure 3

The slopes of the lines AB and AC are $-a/b$ and $-a/c$, hence the slopes of CF and DP are b/a, and the slopes of BE and DS are c/a. It follows that the equations of the various lines in the figure are as follows:

- Equation of AB: $ax + by = ab$
- Equation of AC: $ax + cy = ac$
- Equation of BE: $cx - ay = bc$
- Equation of CF: $bx - ay = bc$
- Equation of DP: $bx - ay = 0$
- Equation of DQ: $ax + cy = 0$
- Equation of DR: $ax + by = 0$
- Equation of DS: $cx - ay = 0$.

Solving appropriate pairs of equations, we get the coordinates of P, Q, R, S:

$$P = \left(\frac{a^2 b}{a^2 + b^2}, \frac{ab^2}{a^2 + b^2} \right),$$
$$S = \left(\frac{a^2 c}{a^2 + c^2}, \frac{ac^2}{a^2 + c^2} \right),$$
$$Q = \left(\frac{bc^2}{a^2 + c^2}, \frac{-abc}{a^2 + c^2} \right),$$
$$R = \left(\frac{b^2 c}{a^2 + b^2}, \frac{-abc}{a^2 + b^2} \right).$$

Now, by a routine calculation (but we omit the details), we find that the slopes of the segments PQ, QR and RS are all equal to the following expression:

$$\frac{a(b+c)}{a^2 - bc}.$$

Hence the points P, Q, R, S lie in a straight line. □

Remarks. The following remarks may be of interest and should be taken note of:

- The coordinates of S may be obtained from the coordinates of P by the switch $b \leftrightarrow c$, i.e., by uniformly switching the roles of b and c.
- Similarly, the coordinates of R may be obtained from the coordinates of Q by the switch $b \leftrightarrow c$, i.e., by uniformly switching the roles of b and c.
- The slope of line $PQRS$ is symmetric in b and c.

If you think about it for a minute, you will realise that each of these observations could have been anticipated before we started the computation. This would have lessened our work.

Some Remarks on Problem-Solving

Let us now critically examine what we have done. What lessons can we draw which will help us in problem-solving in general?

This particular problem viewed as a 'pure geometry problem' is not too difficult; it yields to elementary angle chasing. However, let us consider the matter more generally.

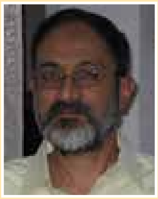
Problems in geometry can sometimes be very challenging, because the figure typically does not give any clue or hint as to the direction in which one must proceed. In general, the difficulty is that one is not able to 'see' the key elements contained in the figure. The solution may depend on seeing that a particular quadrilateral is cyclic; but the quadrilateral may be well-concealed within the figure, its sides not standing out in any way. Or the solution may depend on seeing that two particular angles are equal; but the arms of the angles may not stand out in any way. For such reasons, it helps if we mark the figure suitably, in advance: systematically look for pairs of angles which are equal to one another, and mark them so; systematically look for pairs of segments that are congruent to one another, and mark them so; similarly, mark pairs of lines which are parallel to one another, or perpendicular to one another, and mark them so; and so on. The judicious use of colour can help in carrying out these markings. Obviously, all these steps by themselves will not guarantee anything; but they can ease the path for us. Often they do, so it is worth taking these steps.

The use of coordinates to solve problems in geometry is generally not recommended (particularly by the problem-solving aficionado who would like to see problems in geometry solved by the methods of pure geometry); the purist tells us: "such approaches should be tried only when other approaches have failed." One reason for their asserting this is that the coordinate geometry approach when opted for too easily can start to leach away at our geometrical intuition. This happens because coordinate geometry is highly algebraic, and the symbols used tend to be driven by their own logic; the driver is the machinery of algebra and not our visual sense, and intuition tends to play a much diminished role. In consequence, our intuitive abilities can start to lose their muscle.

Though there is certainly some truth in the above comments, it is important to realise the nuances involved. For example, it is not true to say that no intuitive feel is involved in the application of coordinates. A very important first step when one uses coordinates is the choice of axes; they must be chosen in such a way as to minimise the number of symbols being used, and exploit to the maximum the symmetries implicit in the figure. In the above proof, note how D was made the origin of the coordinate system, with BC and AD as the axes; this has clearly been done keeping in mind the number of lines of the figure which pass through D. Next, note the symbols used for the coordinates of B and C, namely: $(b, 0)$ and $(c, 0)$. The beginner, noting that in the figure B lies to the 'left' or negative side of D while C lies to the 'right' or positive side of D, may be tempted to write: $B = (-b, 0)$ and $C = (c, 0)$. But this would be quite unnecessary, because b and c can be either positive or negative; no signs have been fixed as yet. Also, such a use of symbols would have spoiled the symmetry which we see at present. Writing $B = (-b, 0)$ would have been contrary to the very spirit of algebra. During the course of the solution we remarked: "The coordinates of S may be obtained from those of P by switching the roles of b and c." But this would not have been possible had we written $B = (-b, 0)$.

Another noteworthy point is the following. A proof based on purely geometrical considerations typically starts by drawing a diagram and marking various relationships on it. Now it is obvious that in drawing a diagram, one is going to end up with some angle larger than some other angle, the triangle may be acute-angled or obtuse-angled (one of them), and so on; without any particular intention in mind, one ends up making certain choices in drawing a diagram, perhaps without even being aware that one has made choices. A proof based on such a diagram may have the obvious defect that certain relationships which are true for the diagram may not be true for the diagram drawn in a different way. For example, in the pure geometry proof given above, we wrote at one point, with reference to Figure 3: Since quadrilateral $QDRH$ is cyclic, it follows that $\angle RQD = \angle RHD$. But a quick glance at Figure 2 (the obtuse-angled case) will show that the quadrilateral is $QDHR$ and not $QDRH$, and in this configuration $\angle RQD$ and $\angle RHD$ are supplementary and not equal to each other! This means that a diagram-based proof may have errors which we do not even suspect! In contrast, a proof based on the use of coordinates is typically diagram-independent and suffers from no such defect.

In summary, one may say that one must pause before applying the method of coordinates and choose the axes in a wise manner, exploiting to the maximum all the symmetries of the figure. And if one opts for a pure geometry approach, it is wise to check whether the reasoning one uses is valid for all possible diagrams.



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