A 80-80-20 Triangle

ClassRoom

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In the previous issue of *At Right Angles*, we studied a geometrical problem concerning the triangle with angles of 130°, 20° and 30°. We made the comment that the problem belongs to a class of geometrical problems dealing with triangles with numerous lines drawn within them, intersecting at angles whose measures are an integer number of degrees; we are required to find the measure of some indicated angle. In this note, we study another problem of this genre—a particularly famous such problem. We present a trigonometric solution as well as a 'pure geometry' solution.

Problem. In Figure 1 we see $\triangle ABC$ with $\measuredangle A = 20^\circ$, $\measuredangle B = 80^\circ = \measuredangle C$. Points E and F are located on sides AC and AB by drawing rays from B and C, such that $\measuredangle EBC = 60^\circ$ and $\measuredangle FCB = 50^\circ$. Segment EF is then drawn. We are asked to find the measure of $\measuredangle BEF$.

Trigonometric solution

Numerous trigonometric identities are going to be used in the solution presented below. Please refer to page 70 of the March 2016 issue of AtRiA for a list of these identities.

Keywords: *Integer degree, sine rule, addition formula for sine, difference formula for cosine, isosceles, equilateral, triangle*

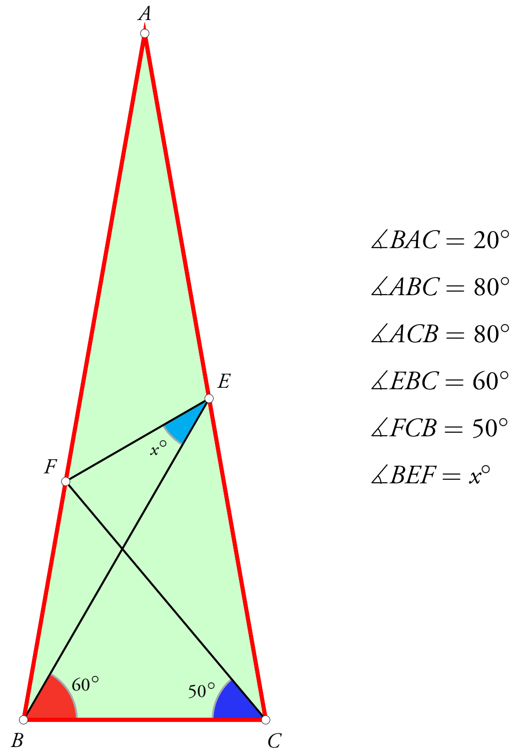

Figure 1

A DIY Invitation!

We can find $\measuredangle BEF$ using the sine rule and the cosine formula for the difference of two angles! See if you can Do-It-Yourself!

Let $\measuredangle BEF = x^\circ$. From $\triangle BEF$ (see Figure 1) we have:

$$\frac{BE}{BF} = \frac{\sin \measuredangle BFE}{\sin \measuredangle BEF} = \frac{\sin(x+20)^\circ}{\sin x^\circ} = \cos 20^\circ + \sin 20^\circ \cdot \cot x^\circ.$$

Also, from $\triangle BCE$ we have:

$$\frac{BE}{BC} = \frac{\sin 80^\circ}{\sin 40^\circ} = 2\cos 40^\circ.$$

Now we have $BC = BF$ (from $\triangle BCF$, in which $\measuredangle BCF = \measuredangle BFC = 50^\circ$). Hence:

$$\cos 20^\circ + \sin 20^\circ \cdot \cot x^\circ = 2\cos 40^\circ,$$

and from this equation we must find x. We have:

$$\begin{aligned}
\sin 20^\circ \cdot \cot x^\circ &= 2\cos 40^\circ - \cos 20^\circ \\
&= \cos 40^\circ + \left(\cos 40^\circ - \cos 20^\circ\right) \\
&= \cos 40^\circ - 2\sin 30^\circ \cdot \sin 10^\circ \\
&= \cos 40^\circ - \sin 10^\circ = \cos 40^\circ - \cos 80^\circ \\
&= 2\sin 20^\circ \cdot \sin 60^\circ = \sqrt{3} \cdot \sin 20^\circ.
\end{aligned}$$

It follows that

$$\cot x^\circ = \sqrt{3},$$

and therefore that $x = 30$. Hence $\measuredangle BEF = 30^\circ$. $\square$

A ‘pure geometry’ solution

The fact that we have obtained such a neat answer to the problem (30°; what could be neater?) challenges us to find a solution to the problem that does not involve computation; in other words, a pure geometry solution. Let us now take up this challenge.

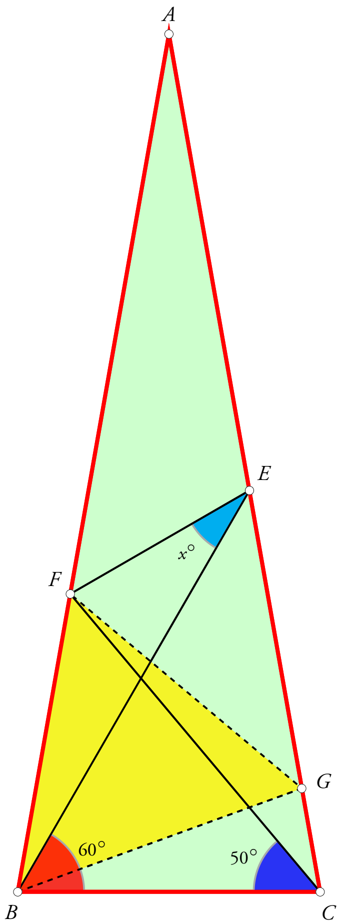

Figure 2

$\measuredangle BAC = 20°$

$\measuredangle ABC = 80°$

$\measuredangle ACB = 80°$

$\measuredangle EBC = 60°$

$\measuredangle FCB = 50°$

$\measuredangle BEF = x°$

Construction: Locate a point G on side AC such that $\measuredangle GBC = 20°$ (so $\measuredangle GBF = 60°$); join GF.

Since $BG = BC$ (this is so because $\measuredangle BCG = 80° = \measuredangle BGC$), and also $BF = BC$, it follows that $BF = BG$; and since $\measuredangle FBG = 60°$, it further follows that $\triangle BFG$ is equilateral. Hence $\measuredangle BGF = 60°$ and $\measuredangle EGF = 40°$.

Next, $GE = GB$, since $\measuredangle GBE = 40° = \measuredangle GEB$; hence $GE = GF$.

It follows that $\measuredangle GEF = \measuredangle GFE = 70°$. Therefore $\measuredangle BEF = 30°$. □

Remark 1. The solution presented above is only one of many pure geometry solutions of this justly famous problem. We invite you to look for one of your own!

Remark 2. Note the important (indeed, crucial) roles played by the equilateral triangle in the above solution and in the pure geometry solution for the 20-30-130 triangle problem (discussed elsewhere in this issue). You will find that this is a recurring theme in almost all such problems.