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# Introducing Differential Calculus on a Graphics Calculator

Barry Kissane

Calculus is fundamentally concerned with understanding and measuring change, which is why it has proved to be such a useful tool for more than three hundred years and has frequently been studied at the end of secondary school. The concept of a derivative is critical to the study of calculus, and is concerned with how functions are changing. In this article, we will outline how a modern graphics calculator can be used to explore this idea.

We will use a particular graphics calculator, the CASIO fx-CG 20, which does not have computer algebra capabilities. Too frequently, students focus on the symbolic manipulation aspects of calculus, which are appropriate for a later and more general treatment of ideas, but not as helpful for an introduction. Of course, similar explorations can be undertaken with other graphics calculators and with various kinds of computer software.

***Keywords:*** *graphics calculator, graph, slope, linear, quadratic, derivative, function*

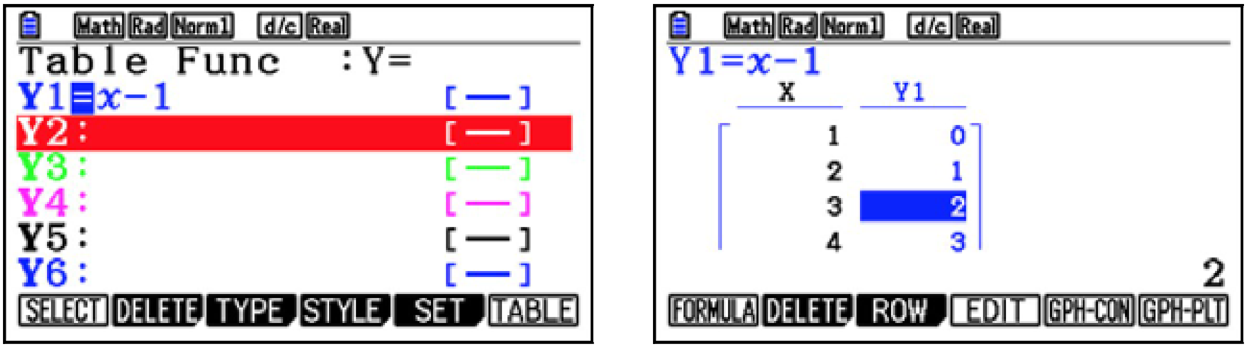


Figure 1

The choice of a graphics calculator is deliberate: it is the only technological tool that has been designed expressly for secondary school mathematics, and so includes a great deal of mathematical functionality designed for education. Kissane and Kemp (2008) described some affordances of graphics calculators for students learning calculus in Australian secondary schools. Graphics calculators have been in use in schools in many countries now for almost thirty years, and yet are frequently misunderstood in other countries (usually by those who have not used them) as devices merely useful to undertake calculations, rather than as educational tools. In many countries that measure student learning with examinations (such as Australia and the USA), students are expected to use their graphics calculators both for everyday learning of mathematics as well as for routine use in examinations. In the case of India, students enrolled in the International Baccalaureate are expected to use graphics calculators as an integral part of their learning, and are also expected to make use of them in official examinations.

Recently, Kissane and Kemp (2014) outlined a model for learning with calculators, suggesting that they can be used to represent mathematical ideas, to undertake computations, to explore mathematical situations and to affirm (or contradict) one's thinking. Of course, the graphics calculator is of value in many other parts of the mathematics curriculum as well, not only for the calculus, as illustrated in Kissane and Kemp (2014). In this article, examples of all of these aspects of calculator use for the particular context of calculus are provided.

**Examining change**

To see how a function is changing, it is convenient to draw a table of values and examine successive terms. For example, consider the simple case of a linear function such as $f(x) = x - 1$. A table of values (see Figure 1) shows that an increase of 1 in $x$ results in an increase of 1 in the value of the function, $f(x)$. Each row of the table, which can be explored with a cursor, shows this steady increase.

Graphics calculators represent functions in three different ways, sometimes referred to as the 'Rule of Three': symbolically, numerically and graphically. Much is to be gained from students toggling between these representations. In this case, if the function is represented graphically, rather than numerically, a familiar linear function, with a steady increase (called the slope, in this case 1) is seen. The screen in Figure 2 shows that the graph can be explored by tracing, which helps to see the connection between the table of values and the graph.

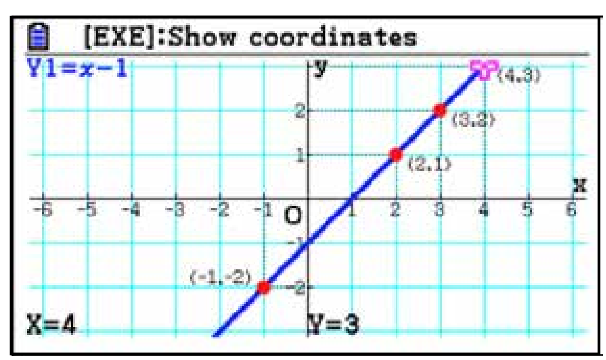


Figure 2

Linear functions are important, as they describe the most fundamental form of growth of a function. Students usually study these, and develop an understanding of slope, before they begin studying the calculus. Both negative and zero slopes and the meaning of larger and smaller slopes are readily explored. Thus, the screen in Figure 3 shows the function $f(x) = 3 - 2x$, which has a slope of -2. The change is still steady, but the values of the function now decrease instead of increasing and change by twice as much as the *x*-values.

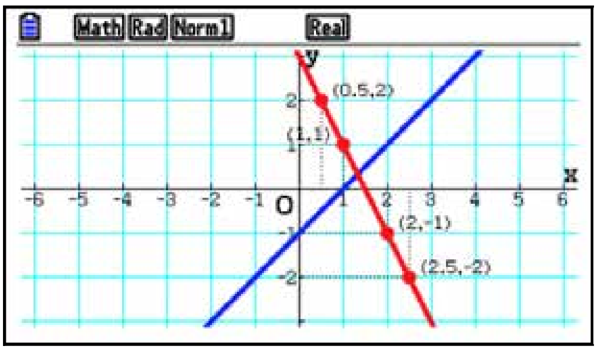


Figure 3

Most functions do not change in steady and easily predictable ways like this, however, and their graphs are not lines but are curves. A good example is the quadratic function, $f(x) = x^2$. Figure 4 shows a curve and not a line, while a table of values shows that the increase in the values of the function is not uniform. An increase in x from 0 to 1 results in an increase of 1 in $f(x)$, but the increase of x from 1 to 2 shows a much larger increase of 3 in $f(x)$. Furthermore, an increase in $x$ from -2 to -1 results in a decrease in $f(x)$.

While describing how linear functions change is relatively accessible, and regularly studied by younger students, this example illustrates that it is clearly a more difficult and complicated matter to describe how non-linear functions change.

**Local linearity**

A very important concept in introductory calculus is local linearity: the idea that, when examined on a small enough (i.e., local) scale, most functions of interest to secondary school students become (almost) linear. This surprising and powerful result is difficult to see without using some technology, which is why it was not commonly discussed in introducing calculus many years ago. Yet it is relatively easy to see on a graphics calculator through a process of 'zooming' both on a graph or on a table of values. A graphics calculator typically has a zoom command, giving an efficient way for a small part of the graph to be seen more closely. The two screens in Figure 5 show this around the

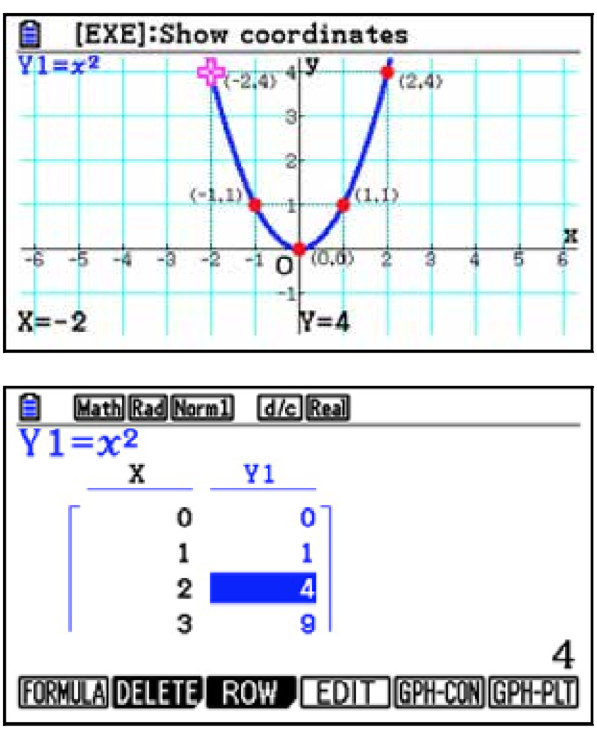


Figure 4

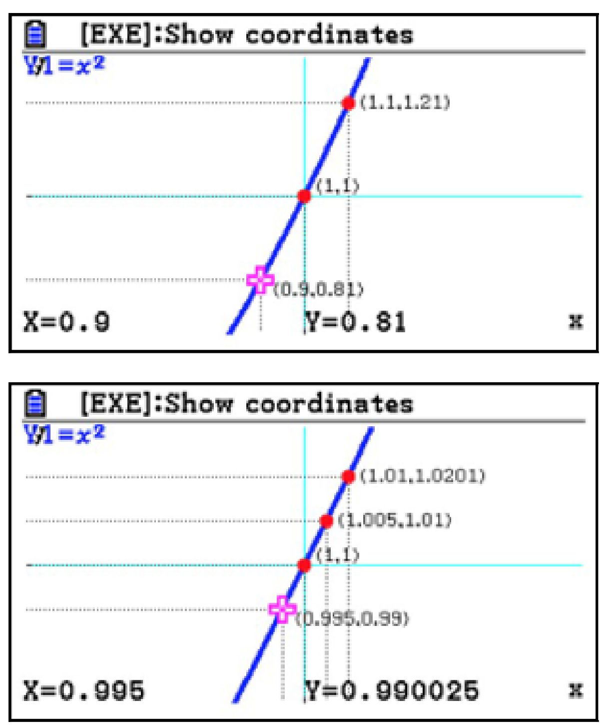


Figure 5

point (1,1) for the graph of $f(x) = x^2$. In each case, the graph that originally looked to be a curve looks (approximately) like a line:

In this case, the second graph has been zoomed in much more than the first graph, and the linearity is more evident. Of course, this process can be continued.

A table of values similarly can be zoomed by choosing a smaller step in the table, as shown in figure 6.

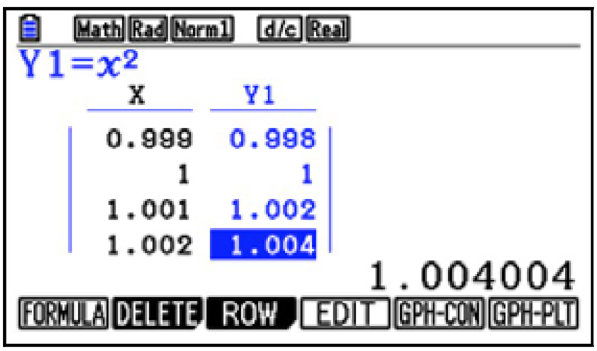


Figure 6

Such exploratory work serves two important purposes. Most importantly, it illustrates that, if examined on a small enough interval, even graphs that are clearly curved and not linear seem to be (almost) linear. Secondly, the idea of the slope of a graph can be seen to be still useful to describe change in a non-linear function, just as it was for linear functions. In this case, for example, it seems that, around the point where $x = 1$, the slope of the function is very close to 2. The table shows this clearly, as an increase of 0.001 in $x$ seems to lead to an increase of twice as much, or about 0.002. In other words, the graph of $f(x) = x^2$ has an approximate slope of 2 when $x$ is close to 1. Further zooming will increase the appearance of linearity and allow the slope of the function to be seen more precisely.

These explorations can of course be undertaken at different points (such as near $x = 2$) or with different functions (such as $f(x) = x^2 - x$.)

**A derivative function**

The idea of determining the slope of a function at different points as a means of describing its change is a powerful – and fundamental – one, so it is not surprising that the calculator has a command to do this automatically. A numerical derivative function, represented by the conventional symbol dy/dx allows students to determine the slope at any point of a graph when tracing. Once this is turned on by the calculator user, and a function explored either graphically or numerically, a value that represents the gradient of the function at each point graphed or tabulated is shown. There are examples of this shown for $f(x) = x^2$ in Figure 7 for both representations:

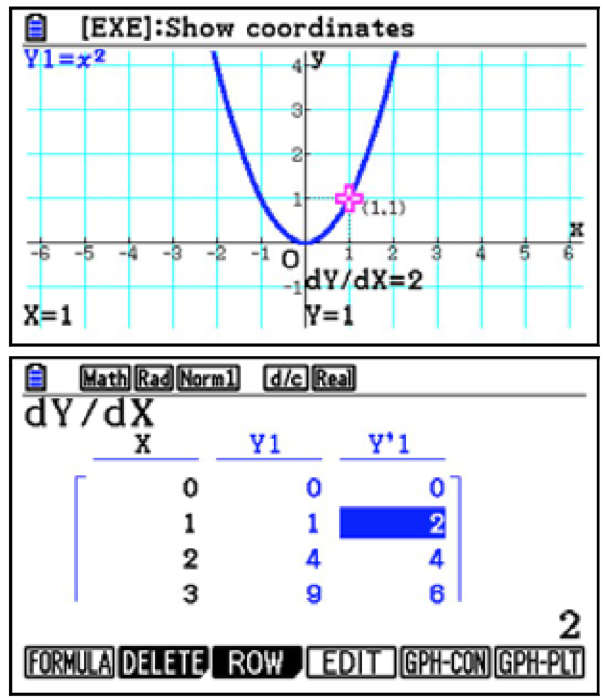


Figure 7

This facility offers strong opportunities for a student to explore what is happening. For example, if the graph is traced and the values of the numerical derivative observed, these are easily seem to be negative (but getting less so) as $x$ increases from -2 to 0 and positive (and getting more so) as $x$ increases from 0 to 2. The value of dy/dx at $x = 0$ is seen to be zero, as for a line of zero slope.

Similarly, the table shows such changes in slope, but also provides an opportunity for students to see that there is a strong relationship apparent between the value of the variable $x$ and the value of the numerical derivative at that point: the derivative is twice the $x$ value, which seems to be

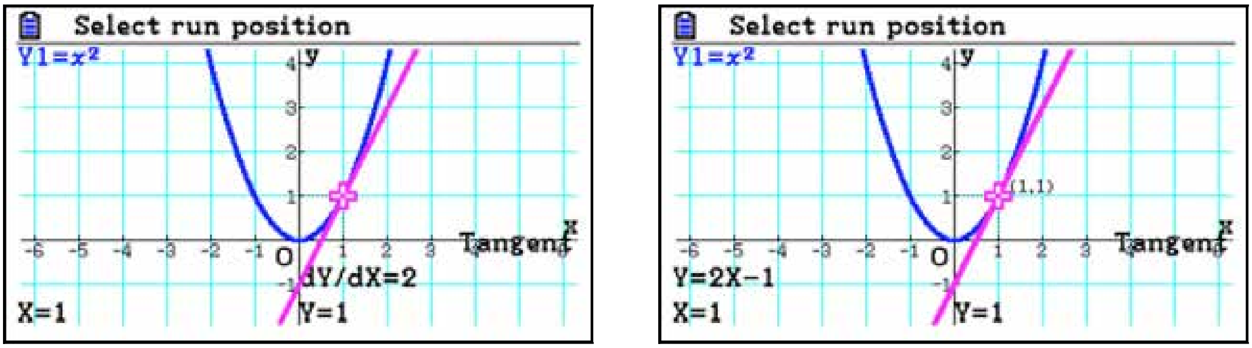


Figure 8

naturally represented as $dy/dx = 2x$ for every value of $x$. This is the crucial idea of a derivative *function*, which describes the derivative at every point in a single relationship. The derivative of a linear function is easily described, since it is the same at every point: the example earlier has $dy/dx = 1$. The relationship for the derivative of a nonlinear function is more complicated, in the sense that it also involves the variable, as the gradient is different in different places.

This approach does not use formal limit proofs and abstract derivations, but is focused on providing a meaning for the idea of both the derivative at a point and a derivative function, which do not require excessive symbolism to make the ideas clear. (It is worth recalling that Newton and Leibniz, co-inventors of the calculus, and Archimedes before them, also did not use the formal mathematical ideas of limits to create meanings for basic calculus ideas.)

**Tangents and derivatives**

A common approach to introductory calculus involves the idea of a secant, becoming closer and closer to a tangent at a point on the curve. While this can also be revealing, it runs a significant risk of confusing the idea of the gradient of a function at a point with the *separate* idea of the gradient of a tangent to a curve at a point. So, it is not the author's preferred way of introducing the idea of a derivative; it seems more conceptually sound to describe the gradient of the function rather than the gradient of a tangent to the function. Nonetheless, a graphics calculator such as the CASIO fx-CG 20 allows for students and teachers interested in doing so to explore this idea by constructing tangents and examining their derivatives, as shown in Figure 8.

The first screen shows the value of the derivative at $x = 1$, while the second screen shows that this tangent at (1,1) can be represented by the line $y = 2x - 1$.

Students can explore the tangents at different points by tracing to different points, and even create an envelope of tangents by stopping at various points on the graph, as shown in Figure 9 for the function $f(x) = 3 - x^{\wedge 2}$.

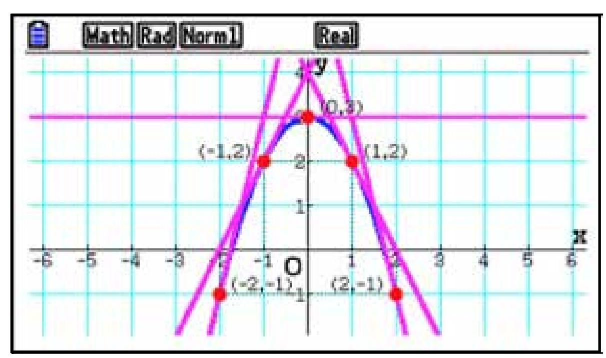


Figure 9

Students who have become familiar with the use of slopes to define functions as decreasing, constant or increasing are likely to find such explorations conceptually helpful.

**Graphing derivative functions**

An especially powerful exploration of the idea of a derivative function is achieved by graphing a function and its derivative function together on a single screen. A graphics calculator allows this to be undertaken by defining one function in terms of another, as shown in Figure 10 for the simple case of $f(x) = x^2$. The second function, Y2, is defined as the (numerical) derivative of the first function.

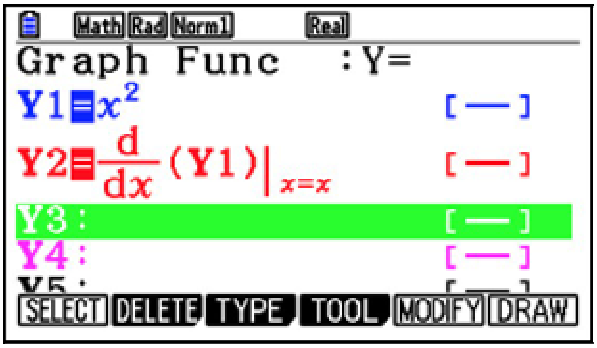


Figure 10

The resulting graph (Figure 11) allows students to see that the derivative function (graphed in red) can be used to describe how the original function (graphed in blue) is changing at various points. In this case, the derivative when $x < 0$ is negative and when $x > 0$ is positive, while there is a zero derivative at $x = 0$ where the red line crosses the $x$-axis.

The derivative is seen to be increasing steadily as $x$ increases, a situation which can be explored more directly with a second derivative function as well, as shown in green in Figure 12. In this case, the second derivative is a constant (2).

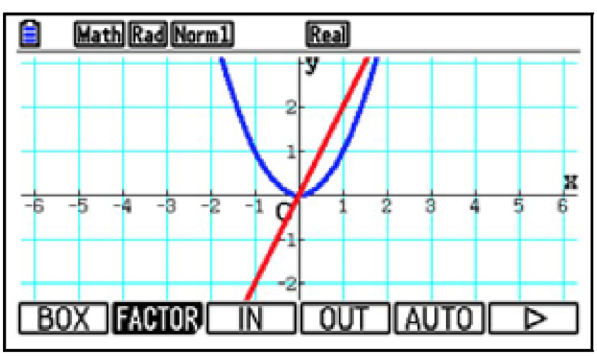


Figure 11

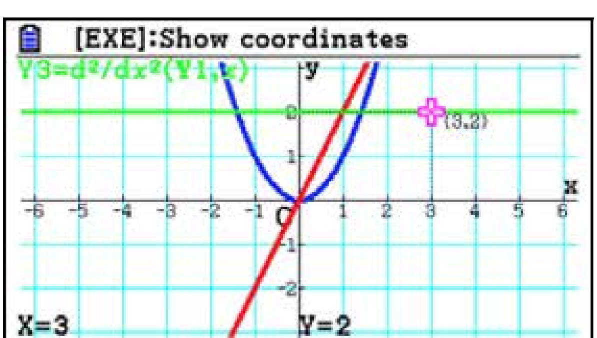


Figure 12

Examining derivatives of other functions is easily undertaken by changing (only) the Y1 function. If students explore quadratic functions in this way, they will see that the derivative functions are always linear, and describe the characteristics of the graph. They will also observe that many functions have the same derivative function – an important understanding needed later when differential equations are studied, helping students to realize that knowing a derivative function is not by itself sufficient to identify the associated function.

Of course, this idea allows a student to explore other kinds of functions than quadratic functions. The graph in Figure 13 of a cubic function $f(x) = x^3 - 3x + 1$ and its derivative shows that the derivative is quadratic and the turning points of the cubic occur where the derivative function crosses the $x$-axis at $x = -1$ and $x = 0$, while there is a point of inflection at the lowest point of the red derivative function ($x = 0$), where the derivative changes from a decreasing to an increasing function.

Explorations like these help to build meaning for the idea of the derivative. Indeed, many interesting class discussions can be undertaken by graphing only

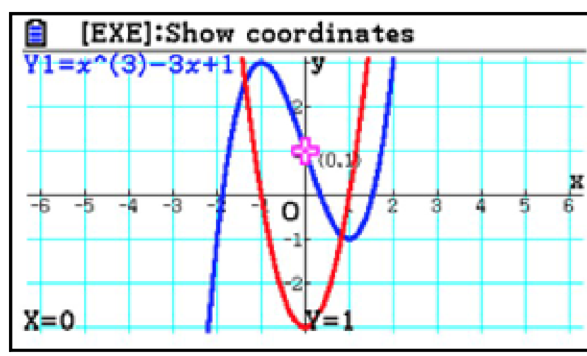


Figure 13

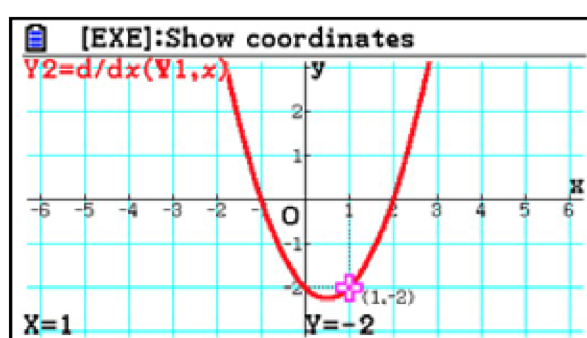


Figure 14

the derivative function and not the actual function, and asking students to describe the properties of the function that has a derivative of that shape. The example in Figure 14 is a case in point:

Students can use the graph of the derivative, which is describing how the function itself is changing, to describe the graph of the function, its two turning points, its point of inflection and its end behaviour for very large or very small values of $x$.

### Derivatives from first principles

Differential calculus is often introduced by defining the derivative from first principles as the limit of the gradient of a secant as the secant gets closer and closer to being a tangent. The idea of 'closer and closer' is a difficult one for students to understand, but good approximations are available on a graphics calculator. We will use a more sophisticated example than previously to illustrate a possible form of exploration: the derivative of the sine function $f(x) = \sin x$, with radian measure being used. The theoretical formulation is usually represented as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In this case, we begin with defining a small value of $h$, say $h = 0.01$. Then the graphics calculator can be used to define the function as Y1 and the quotient as Y2 (Figure 15).

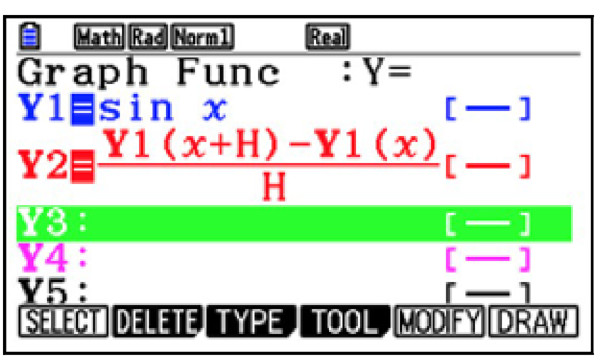


Figure 15

The resulting graph in Figure 16 (on a suitable scale with horizontal grid lines at every $\pi/2$) shows the sine function in blue and the approximate derivative in red. Students studying this situation will be expected to recognize that the red graph looks like that of the cosine function $f(x) = \cos x$, a quite unexpected result (for students, not their teachers).

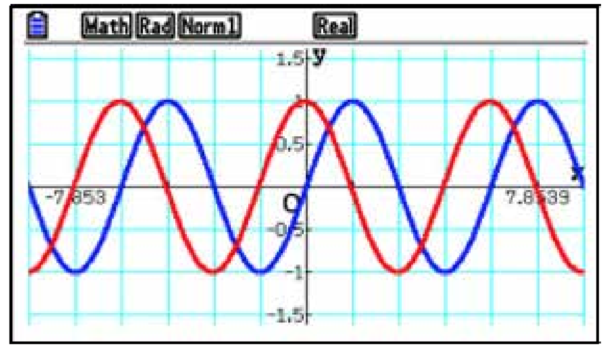


Figure 16

A good way to see whether this prediction is correct is to add a third graph (of the cosine function), as shown in the next screen (Figure 17). The new graph seems very close to the red graph, but they are not quite overlapping, suggesting that they are not quite the same:

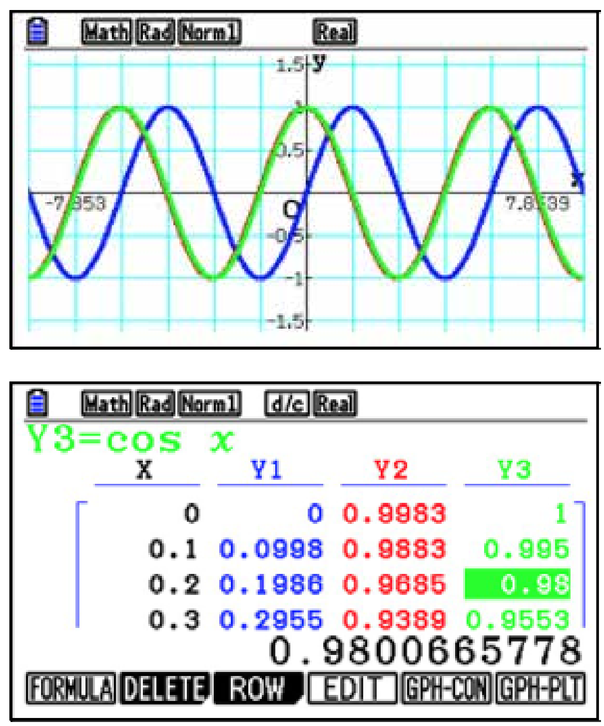


Figure 17

A table of values is also helpful here. Figure 17 shows that the second and third functions are similar, but not identical, which is a timely warning not to be deceived by the appearance of graphs.

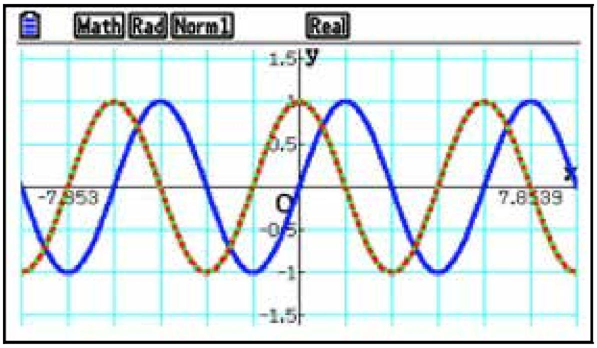


Figure 18

If the value of $h$ is reduced, the idea of 'closer and closer' (and the formal idea of convergence and of a limit) can be further explored. For example, Figure 18 shows $h$ = 0.001, for which the graphs of Y2 and Y3 are indistinguishable to the eye (with the third graph being drawn in a dotted style to show the overlap):

The tabled values are still slightly different, as seen in Figure 19 however, from around the third

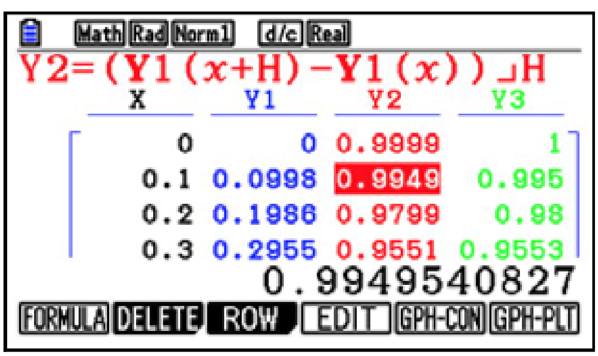


Figure 19

decimal place. Allowing $h$ to be still closer to 0, but still positive will even give a table of values that gives an impression that the second and third functions are identical, as shown below, but a closer inspection in Figure 20 suggests that they are still slightly different.

Students can readily explore this situation and gain a good sense of the meaning of important concepts such as limit and convergence, as well as the derivative of the sine function. Without the use of technology, it is very hard for students to appreciate and understand the subtle ideas involved.

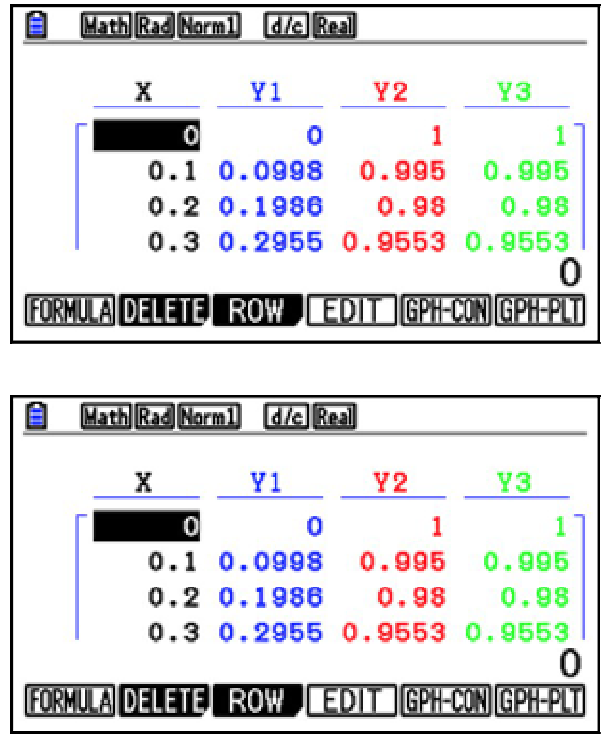


Figure 20

Of course, at some point, most mathematics curricula introduce formal mathematical proofs of results of these kinds, and handle the processes symbolically. However, the graphics calculator offers strong opportunities for this more formal work to be built upon practical experiences and intuitions; indeed, the work with the graphics calculator might even provide a motivation for a more formal and more theoretical development to be provided.

**Conclusion**

As claimed earlier, the graphics calculator is regularly misunderstood as a calculation device instead of its proper role as an educational device being recognized. Similarly, the focus on the graphing capabilities may often mask the importance of numerical approaches to understanding mathematical ideas, in addition to visual ones. The examples offered here show some of the ways in which the inbuilt capabilities of a modern graphics calculator can be used by students and exploited by teachers to help build a solid foundation for important mathematical ideas of differential calculus.

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