Problems for the Senior School

Problem Editors: Prithwijit De & Shailesh Shirali

Problem Corner

PROBLEMS FOR SOLUTION

Call a convex quadrilateral *tangential* if a circle can be drawn tangent to all four sides. In this edition of the problem set, all the problems that we have posed have to do with this notion.

Problem V-2-S.1
Let $ABCD$ be a tangential quadrilateral. Prove that $AD + BC = AB + CD$.

Problem V-2-S.2
Let $ABCD$ be a convex quadrilateral with $AD + BC = AB + CD$. Prove that $ABCD$ is tangential.

Problem V-2-S.3
Place four coins of different sizes on a flat table so that each coin is tangent to two other coins. Prove that the quadrilateral formed by joining the centres of the coins is tangential. Prove also that the convex quadrilateral whose vertices are the four points of contact is cyclic. Is the circle passing through the four points of contact tangent to the sides of the tangential quadrilateral formed by joining the four centres of the coins?

Problem V-2-S.4
Let $ABCD$ be a cyclic quadrilateral and let X be the intersection of diagonals AC and BD. Let P_1, P_2, P_3 and P_4 be the feet of the perpendiculars from X to BC, CD, DA and AB respectively. Prove that quadrilateral $P_1P_2P_3P_4$ is tangential.

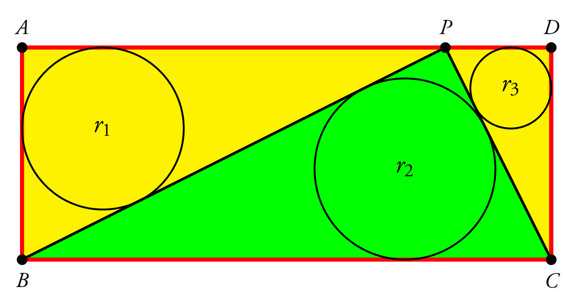
SOLUTIONS OF PROBLEMS IN ISSUE-V-1 (MARCH 2016)

Solution to problem V-1-S.1
What is the greatest possible perimeter of a right-angled triangle with integer sides, if one of the sides has length 12?

We claim that it cannot be the hypotenuse whose length is 12. For if it were, let the other two sides be a, b; then $a^2 + b^2 = 12^2$. Since the RHS is an even number, a and b must be both odd or both even. The first possibility does not work out as it leads to $a^2 \equiv 1 \pmod 4$ and $b^2 \equiv 1 \pmod 4$, implying that $a^2 + b^2 \equiv 2 \pmod 4$; but 12^2 is a multiple of 4. Hence both a and b are even numbers. Let $a = 2a_1$, $b = 2b_1$, where a_1, b_1 are integers. We now get $a_1^2 + b_1^2 = 36$. Repeating the same argument, we see that both a_1 and b_1 are even numbers; say $a_1 = 2a_2$, $b_1 = 2b_2$, where a_2, b_2 are integers. We now get $a_2^2 + b_2^2 = 9$. However, it is easy to check that 9 cannot be expressed as a sum of two perfect squares. Hence it cannot be the hypotenuse whose length is 12.

Let the hypotenuse be x units and the other side y units. Then we have: $x^2 - y^2 = 144$, hence $x + y$ and $x - y$ are factors of 144 and as both are of same parity, both must be even. The perimeter of the triangle is $x + y + 12$. Thus for maximum perimeter we must make $x + y$ as large as possible and hence $x - y$ as small as possible. The least possible value of $x - y$ is 2, so the greatest possible value of $x + y$ is 72; the corresponding perimeter is 84 units. (For the actual dimensions of the triangle: $x + y = 72$, $x - y = 2$, hence $x = 37$, $y = 35$; so the sides of the triangle are 37, 35, 12.)

Solution to problem V-1-S.2
Rectangle ABCD has sides $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\angle BPC = 90^\circ$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC and CPD, what is the value of $r_1 + r_2 + r_3$?



Observe that

$$r_1 = \frac{AP + AB - BP}{2}, \quad r_2 = \frac{BP + PC - BC}{2}, \quad r_3 = \frac{DP + CD - PC}{2}.$$

Adding these we obtain

$$r_1 + r_2 + r_3 = \frac{AD + AB + CD - BC}{2} = AB = 8.$$

Solution to problem V-1-S.3
Let a, b, c be such that $a + b + c = 0$; find the value of P where

$$P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}.$$

Since $a+b+c=0$ we have

$$2a^2+bc=a^2+a^2+bc=a^2-a(b+c)+bc=(a-b)(a-c).$$

So:

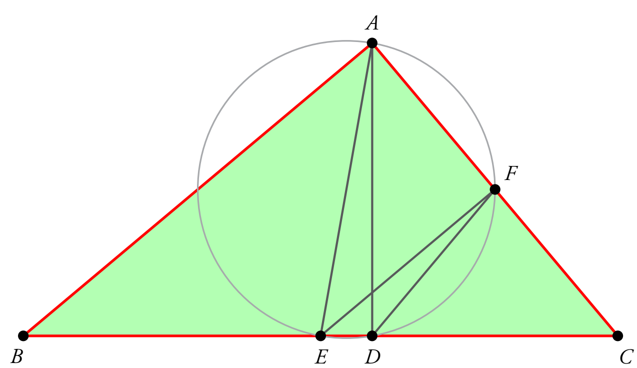
$$\frac{a^2}{2a^2+bc}=\frac{a^2}{(a-b)(a-c)},\quad \frac{b^2}{2b^2+ca}=\frac{b^2}{(b-c)(b-a)},\quad \frac{c^2}{2c^2+ab}=\frac{c^2}{(c-a)(c-b)}.$$

Hence:

$$\begin{aligned} P&=\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}\\ &=-\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{(a-b)(b-c)(c-a)}\\ &=\frac{(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)}=1. \end{aligned}$$

Solution to problem V-1-S.4

In acute-angled triangle ABC, let D be the foot of the altitude from A, and E be the midpoint of BC. Let F be the midpoint of AC. Suppose $\measuredangle BAE=40^\circ$. If $\measuredangle DAE=\measuredangle DFE$, what is the magnitude of $\measuredangle ADF$ in degrees?



Since $EF \parallel AB$, $\measuredangle BAE=\measuredangle AEF=40^\circ$. As $\measuredangle DAE=\measuredangle DFE$, points A, D, E and F are concyclic. Hence $\measuredangle ADF=\measuredangle AEF=40^\circ$.

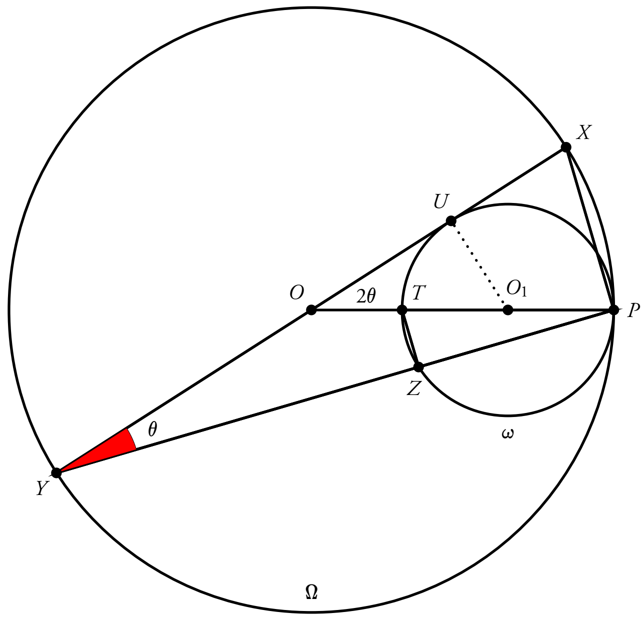
Solution to problem V-1-S.5

Circle ω touches the circle Ω internally at P. The centre O of Ω is outside ω. Let XY be a diameter of Ω which is also tangent to ω. Assume that $PY>PX$. Let PY intersect ω at Z. If $YZ=2PZ$, what is the magnitude of $\measuredangle PYX$ in degrees?

Let T be the point of intersection of OP and ω. Let $\measuredangle PYX=\theta$. Then $\measuredangle OPY=\theta$ and $\measuredangle XOP=2\theta$. As ω is internally tangent to Ω, PT is a diameter of ω.

In $\triangle PZT$ and $\triangle YPX$, we have: $\measuredangle PZT=\measuredangle XPY=90^\circ$ and $\measuredangle TPZ=\measuredangle PYX=\theta$. Hence $\triangle PZT\sim\triangle YPX$. This implies that

$$\frac{PT}{YX}=\frac{PZ}{YP}=\frac{1}{3}.$$



If r and R are the radii of ω and Ω, respectively, then we have

$$\frac{r}{R} = \frac{1}{3}.$$

If O_1 is the centre of ω, then we see that $OO_1 = R - r = 2r$. Let U be the foot of the perpendicular from O_1 to XY. Then

$$\sin 2\theta = \frac{UO_1}{OO_1} = \frac{r}{2r} = \frac{1}{2}.$$

Hence $\theta = 15°$.