A Problem Study

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Problem Corner

We present a close study of a problem posed in the March 2016 issue. In the process, we demonstrate that there may be more to a problem than just the solution of that problem.

Sums of consecutive squares

In the 'Middle Problems' section (March 2016 issue), Problem V-1-M.4 (b) (solved elsewhere in this very issue) asked:

> *Can the sum of* 11 *consecutive perfect squares be a perfect square?*

We shall show that this phenomenon occurs infinitely often. But first we ask a question that will almost certainly have occurred to you. Why 11? Why not 2? Why not 3? Why not 4? …So let us attempt to address this series of questions first.

Can the sum of two consecutive squares be a perfect square?

The answer is 'Yes'. Let the two consecutive squares be n^2 and $(n+1)^2$. Their sum is

$$n^2 + (n+1)^2 = 2n^2 + 2n + 1.$$

Suppose that this is a perfect square, x^2. Then we have:

$$x^2 = 2n^2 + 2n + 1,$$

$$\therefore\ 2x^2 = 4n^2 + 4n + 2 = (2n+1)^2 + 1.$$

Write $y = 2n + 1$; then we have $2x^2 = y^2 + 1$, i.e.,

$$y^2 - 2x^2 = -1, \tag{1}$$

and we have reached an equation which by now should be very familiar to you: the Brahmagupta-Fermat-Pell equation. Note that we want solutions in which x is odd. We will solve equation (1) in an ad hoc way for now. Computer-based trials yield the following x, y values which solve (1). The third row gives the corresponding values of n.
(Recall that $y = 2n + 1$.)

x	1	5	29	169	985	...
y	1	7	41	239	1393	...
n	0	3	20	119	696	...

These figures give rise to the following instances where the sum of the squares of two consecutive positive integers is itself a perfect square:

$$\begin{cases} 3^2 + 4^2 = 5^2, \\ 20^2 + 21^2 = 29^2, \\ 119^2 + 120^2 = 169^2, \\ 696^2 + 697^2 = 985^2, \quad \ldots. \end{cases}$$

As always in such settings, there are numerous patterns to be spotted in the sequences. Thus we have, for the sequence of y-values:

$$41 = 6 \times 7 - 1,$$

$$239 = 6 \times 41 - 7,$$

$$1393 = 6 \times 239 - 41,$$

and so on. That is, if the y-values are $y_1 = 1, y_2 = 7, y_3 = 41, \ldots$, then

$$y_{n+2} = 6y_{n+1} - y_n, \tag{2}$$

and exactly the same recurrence holds for the sequence of x-values:

$$x_{n+2} = 6x_{n+1} - x_n, \tag{3}$$

where $x_1 = 1, x_2 = 5, x_3 = 29, \ldots$.

Another striking recurrence relation which is also computationally very convenient is the following:

$$\begin{cases} x_{n+1} = 3x_n + 2y_n, \\ y_{n+1} = 4x_n + 3y_n. \end{cases} \tag{4}$$

All these relations can be proved using induction. It follows from them that there are infinitely many instances where the sum of the squares of two consecutive numbers is itself a perfect square.

Can the sum of three consecutive squares be a perfect square? Let the 3 consecutive squares be $n^2, (n+1)^2, (n+2)^2$. Their sum is

$$\begin{aligned} n^2 + (n+1)^2 + (n+2)^2 &= 3n^2 + 6n + 5 \\ &= 3(n+1)^2 + 2. \end{aligned} \tag{5}$$

We see that the sum is of the form $3k + 2$, i.e., it leaves remainder 2 under division by 3. However, no perfect square is of this form. (Under division by 3, all squares leave remainder 0 or 1.) Hence the sum of three consecutive squares can never be a perfect square.

Can the sum of four consecutive squares be a perfect square? Let the 4 consecutive squares be $n^2, (n+1)^2, (n+2)^2, (n+3)^2$. Their sum is

$$\begin{aligned} n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 &= 4n^2 \\ + 12n + 14 = 4\left(n^2 + 3n + 3\right) + 2. \end{aligned} \tag{6}$$

We see that the sum is of the form $4k + 2$, i.e., it leaves remainder 2 under division by 4. However, no perfect square is of this form. (Under division by 4, all squares leave remainder 0 or 1.) Hence the sum of four consecutive squares can never be a perfect square.

Can the sum of five consecutive squares be a perfect square? Let the 5 consecutive squares be $n^2, (n+1)^2, \ldots, (n+4)^2$. Their sum is

$$\begin{aligned} \sum_{k=0}^{4} (n+k)^2 &= 5n^2 + 20n + 30 \\ &= 5(n+2)^2 + 10. \end{aligned} \tag{7}$$

Suppose that this quantity is a perfect square, say a^2. Then a^2 is a multiple of 5, hence a itself is a multiple of 5; say $a = 5b$ where b is an integer. This leads to:

$$\begin{aligned} 25b^2 &= 5(n+2)^2 + 10, \\ \therefore\ 5b^2 &= (n+2)^2 + 2. \end{aligned} \tag{8}$$

This equality implies that $(n+2)^2$ leaves remainder 3 under division by 5. However, no perfect square is of this form. (Under division by 5, all squares leave remainder 0, 1 or 4.) Hence the sum of five consecutive squares can never be a perfect square.

Can the sum of six consecutive squares be a perfect square? Let the 6 consecutive squares be $n^2, (n+1)^2, \ldots, (n+5)^2$. Their sum is

$$\sum_{k=0}^{5}(n+k)^2 = 6n^2 + 30n + 55. \qquad (9)$$

Now observe that

$$6n^2 + 30n = 6n(n+5).$$

The quantity $n(n+5)$ is even, as it is the product of an even number, and an odd number. Hence $6n(n+5)$ is a multiple of 4. Since 55 leaves remainder 3 under division by 4, the quantity $6n^2 + 30n + 55$ leaves remainder 3 under division by 4. However, no perfect square is of this form. (Under division by 4, all squares leave remainder 0 or 1.) Hence the sum of six consecutive squares can never be a perfect square.

Can the sum of seven consecutive squares be a perfect square? Let the 7 consecutive squares be $n^2, (n+1)^2, \ldots, (n+6)^2$. Their sum is

$$\begin{aligned}\sum_{k=0}^{6}(n+k)^2 &= 7n^2 + 42n + 91\\ &= 7\left(n^2 + 6n + 13\right). \qquad (10)\end{aligned}$$

Suppose that this quantity is a perfect square, say a^2. Then a^2 is a multiple of 7, hence a itself is a multiple of 7; say $a = 7b$ where b is an integer. This leads to:

$$\begin{aligned}49b^2 &= 7\left(n^2 + 6n + 13\right),\\ \therefore\ 7b^2 &= (n+3)^2 + 4. \qquad (11)\end{aligned}$$

This equality implies that $(n+3)^2$ leaves remainder 3 under division by 7. However, no perfect square is of this form. (Under division by 7, all squares leave remainder 0, 1, 2 or 4.) Hence the sum of seven consecutive squares can never be a perfect square.

Can the sum of eight consecutive squares be a perfect square? Let the 8 consecutive squares be $n^2, (n+1)^2, \ldots, (n+7)^2$. Their sum is

$$\begin{aligned}\sum_{k=0}^{7}(n+k)^2 &= 8n^2 + 56n + 140\\ &= 8n(n+7) + 140. \qquad (12)\end{aligned}$$

The quantity $n(n+7)$ is even, as it is the product of an even number, and an odd number. Hence $8n(n+7)$ is a multiple of 16. Since 140 leaves remainder 12 under division by 16, the quantity $8n^2 + 56n + 140$ leaves remainder 12 under division by 16. However, no perfect square is of this form. (Under division by 16, all squares leave remainder 0, 1, 4 or 9.) Hence the sum of eight consecutive squares can never be a perfect square.

Can the sum of nine consecutive squares be a perfect square? Let the 9 consecutive squares be $n^2, (n+1)^2, \ldots, (n+8)^2$. Their sum is

$$\begin{aligned}\sum_{k=0}^{8}(n+k)^2 &= 9n^2 + 72n + 204\\ &= 9\left(n^2 + 8n + 22\right) + 6. \qquad (13)\end{aligned}$$

Hence the quantity $9n^2 + 72n + 204$ leaves remainder 6 under division by 9. However, no perfect square is of this form. (Under division by 9, all squares leave remainder 0, 1, 4 or 7.) Hence the sum of nine consecutive squares can never be a perfect square.

Can the sum of ten consecutive squares be a perfect square? Let the 10 consecutive squares be $n^2, (n+1)^2, \ldots, (n+9)^2$. Their sum is

$$\sum_{k=0}^{9}(n+k)^2 = 10n^2 + 90n + 285. \qquad (14)$$

Suppose that this quantity is a perfect square, say a^2. Then a^2 is a multiple of 5, hence a itself is a multiple of 5; say $a = 5b$ where b is an integer. This leads to:

$$\begin{aligned}25b^2 &= 10n^2 + 90n + 285,\\ \therefore\ 10b^2 &= 4n^2 + 36n + 114, \qquad (15)\end{aligned}$$

and so:

$$10b^2 = (2n+9)^2 + 33. \qquad (16)$$

Since the quantity on the left side of this equality is a multiple of 10, it must be that $(2n+9)^2$ leaves remainder 7 under division by 10. However, no perfect square is of this form. (Under division by 10, all squares leave remainder 0, 1, 4, 5, 6 or 9.) Hence the sum of ten consecutive squares can never be a perfect square.

Remark. We have found something quite remarkable. A sum of 2 consecutive squares can be a perfect square, but not a sum of 3 or 4 or 5 or 6 or 7 or 8 or 9 or 10 consecutive squares! This phenomenon now prepares us to believe that a sum of 11 consecutive squares cannot be a perfect square either. But wait and see ....

Can the sum of eleven consecutive squares be a perfect square? Let the 11 consecutive squares be $n^2, (n+1)^2, (n+2)^2, \ldots, (n+10)^2$. Their sum is

$$\sum_{k=0}^{10}(n+k)^2 = 11n^2 + 110n + 385$$
$$= 11\left(n^2 + 10n + 35\right). \qquad (17)$$

Can this quantity be a perfect square? On this occasion, we shall not attempt a proof of impossibility—because a computer assisted search does uncover some solutions! We will not reveal this solution for now, but wait for the analysis to uncover it on its own.

For $\sum_{k=0}^{10}(n+k)^2$ to be a square, $n^2 + 10n + 35$ must have the form $11m^2$ for some integer m. And since $n^2 + 10n + 35 = (n+5)^2 + 10$, this demand leads to the following equation (after the substitution $x = n + 5$):

$$x^2 - 11m^2 = -10. \qquad (18)$$

We shall solve this equation (and show the existence of infinitely many solutions in positive integers) in three stages:

Step 1: Find a single solution (x_0, m_0) to the equation, using trial and error. (Use computer experimentation if necessary.)

Step 2: Generate infinitely many solutions (in positive integers) to the 'auxiliary equation' $x^2 - 11m^2 = 1$. (There is a standard method for doing this, as described below.)

Step 3: Use the solution (x_0, m_0) and the infinite family of solutions to the auxiliary equation $x^2 - 11m^2 = 1$ to generate infinitely many solutions in positive integers to the original equation $x^2 - 11m^2 = -10$. (We need to 'compose' the two solutions with one another; the method is described below.)

Step 1 is trivial, because $x = 1, m = 1$ solves the equation; so we put $x_0 = 1$, $m_0 = 1$. (We can count ourselves lucky that this solution can be obtained by inspection, without any need to use computing machinery.)

For Step 2 we use a standard algorithm which works for all such equations. For now, we shall not discuss the theory behind this algorithm, but only remark that it works. We start by finding the simple continued fraction (SCF) for $\sqrt{11}$. Observe that:

$$\sqrt{11} = 3 + \left(\sqrt{11} - 3\right) = 3 + \frac{2}{\sqrt{11}+3}$$
$$= 3 + \frac{1}{(\sqrt{11}+3)/2}$$
$$= 3 + \frac{1}{3 + (\sqrt{11}-3)/2}$$
$$= 3 + \frac{1}{3 + 1/(\sqrt{11}+3)}$$
$$= 3 + \cfrac{1}{3 + \cfrac{1}{6 + (\sqrt{11}-3)}}.$$

Noting that the form $\sqrt{11} - 3$ has occurred again, we deduce that the SCF for $\sqrt{11}$ is periodic:

$$\sqrt{11} = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \cdots}}}}. \qquad (19)$$

This is generally written in the following form:

$$\sqrt{11} = [3; 3, 6, 3, 6, 3, 6, 3, 6, \ldots], \qquad (20)$$

with the 3, 6 repeating infinitely often. A convenient short form for this is to use a bar above the 3, 6 (just as we do for recurring decimals):

$$\sqrt{11} = [3; \overline{3, 6}], \qquad (21)$$

Now we compute the convergents to the continued fraction, i.e., the fractions obtained by truncating its tail:

$$[3] = 3,$$

$$[3;3] = 3 + \frac{1}{3} = \frac{10}{3},$$

$$[3;3,6] = 3 + \cfrac{1}{3 + \cfrac{1}{6}} = \frac{63}{19},$$

$$[3;3,6;3] = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3}}} = \frac{199}{60},$$

$$[3;3,6;3] = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6}}}} = \frac{1257}{379},$$

and so on. The solution pairs to $x^2 - 11m^2 = 1$ in positive integers are now:

$$(x, m) = (10, 3),\ (199, 60),\ \ldots. \tag{22}$$

All these solutions can also be generated from the very first one, by the following very simple procedure:

$$\left(10 + 3\sqrt{11}\right)^2 = 199 + 60\sqrt{11},$$

$$\left(10 + 3\sqrt{11}\right)^3 = 3970 + 1197\sqrt{11},$$

$$\left(10 + 3\sqrt{11}\right)^4 = 79201 + 23880\sqrt{11},$$

and so on. The solution pairs to $x^2 - 11m^2 = 1$ in positive integers are thus:

$$(x, m) = (10, 3),\ (199, 60),\ (3970, 1197),\ (79201, 23880),\ \ldots. \tag{23}$$

For Step 3 we compose the pair $(x_0, m_0) = (1, 1)$ with the above solution pairs via the process of multiplication by $10 + 3\sqrt{11}$:

$$\left(1 + \sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)^0 = 1 + 1\sqrt{11},$$

$$\left(1 + \sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)^1 = 43 + 13\sqrt{11},$$

$$\left(1 + \sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)^2 = 859 + 259\sqrt{11},$$

$$\left(1 + \sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)^3 = 17137 + 5167\sqrt{11},$$

$$\left(1 + \sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)^4 = 341881 + 103081\sqrt{11},$$

and so on; we thus get the following solution pairs to the equation $x^2 - 11m^2 = -10$:

$$(x, m) = (1, 1),\ (43, 13),\ (859, 259),\ (17137, 5167),\ (341881, 103081),\ \ldots, \tag{24}$$

and this list is infinite. Since

$$\left(a + b\sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right) = (10a + 33b) + (3a + 10b)\sqrt{11}, \tag{25}$$

the pairs listed can also be generated by the following recursive procedure: $x_1 = 1$, $m_1 = 1$, and for $k \geq 1$:

$$x_{k+1} = 10x_k + 33m_k,$$
$$m_{k+1} = 3x_k + 10m_k. \tag{26}$$

The procedures we have described clearly generate infinitely many solution pairs to the equation $x^2 - 11m^2 = -10$ and thus provide infinitely many instances to the question posed at the beginning: *How many instances are there when the sum of* 11 *consecutive perfect squares is itself a perfect square?* Thus we have:

$$38^2 + 39^2 + \cdots + 48^2 = 20449 = 143^2,$$

$$854^2 + 855^2 + \cdots + 864^2 = 8116801 = 2849^2,$$

$$17132^2 + 17133^2 + \cdots + 17142^2 = 3230444569 = 56837^2,$$

and so on.

A second family of solutions. Another infinite family of solutions gets generated if we start with a different solution in Step 1; namely: $x_0 = -1$, $m_0 = 1$. We get:

$$\left(-1+\sqrt{11}\right)\cdot\left(10+3\sqrt{11}\right)^0 = -1+1\sqrt{11},$$

$$\left(-1+\sqrt{11}\right)\cdot\left(10+3\sqrt{11}\right)^1 = 23+7\sqrt{11},$$

$$\left(-1+\sqrt{11}\right)\cdot\left(10+3\sqrt{11}\right)^2 = 461+139\sqrt{11},$$

$$\left(-1+\sqrt{11}\right)\cdot\left(10+3\sqrt{11}\right)^3 = 9197+2773\sqrt{11},$$

$$\left(-1+\sqrt{11}\right)\cdot\left(10+3\sqrt{11}\right)^4 = 183479+55321\sqrt{11},$$

and so on; we thus get the following solution pairs to the equation $x^2 - 11m^2 = -10$:

$$(x, m) = (23, 7),\ (461, 139),\ (9197, 2773),\ (183479, 55321),\ \ldots, \tag{27}$$

and this list too is infinite. These pairs yield the following equalities:

$$18^2 + 19^2 + \cdots + 28^2 = 5929 = 77^2,$$

$$456^2 + 457^2 + \cdots + 466^2 = 2337841 = 1529^2,$$

$$9192^2 + 9193^2 + \cdots + 9202^2 = 930433009 = 30503^2,$$

and so on.

Closing Remarks

A natural follow-up to the above is to ask whether the sum of 12 consecutive squares can be a perfect square; and likewise for 13, 14, 15, …consecutive squares. It would be quite difficult to anticipate the results in advance, i.e., in which cases we do obtain solutions, and which cases we do not. But we leave this analysis for you to take forward.

In closing, we add a few remarks amplifying on the comment made at the start: that there may be more to a problem than just the solution of the problem.

In his famous book *How To Solve It*, George Pólya suggests the following four key steps in problem solving: (i) understand the problem; (ii) devise a plan; (iii) carry out the plan; (iv) review/extend the solution and analysis. Our interest in this section is in suggestion (iv). There are two aspects to what Pólya has said: *reviewing* what one has done, which means taking the trouble to look back at one's effort and to identify the steps that worked and the steps that did not work; this little extra step can contribute significantly to one's 'problem-solving muscle.' The other aspect is: *extending one's work*. There is no set approach for doing this, but one can start by tweaking the problem, altering the numbers or the initial conditions and examining the consequences of these tweaks. Playing around in this manner with the problem, one can actually uncover fresh and new mathematics. In that sense, it can be a wonderfully enriching activity, one which every student and every mathematics teacher must experience. This is just what we have done in our article, and we hope that it has proved worthwhile for you to read it.

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