Adventures with Triples

Part I

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Problem Corner

In this two-part note, we study the following two problems. Given three positive integers a, b, c, we say that the triple (a, b, c) has the **linear property** if the sum of some two of the three numbers equals the third number (i.e., either $a + b = c$ or $b + c = a$ or $c + a = b$); and we say that the triple has the **triangular property** if the sum of any two of the three numbers exceeds the third number (i.e., $a + b > c$ and $b + c > a$ and $c + a > b$).

Fix any upper limit n, and let a, b, c take all possible positive integer values between 1 and n (i.e., $1 \leq a, b, c \leq n$). There are clearly n^3 such triples. How many of these triples possess the linear property? How many of these triples possess the triangular property? These are the questions that we plan to explore.

Notation

- $S(n)$ denotes the set of all triples (a, b, c) with $1 \leq a, b, c \leq n$. The number of such triples is clearly n^3.
- $L(n)$ denotes the number of triples in $S(n)$ which possess the linear property.
- $T(n)$ denotes the number of triples in $S(n)$ which possess the triangular property.

In Part I of the article, we shall focus on computing $L(n)$.

Keywords: *Integers, linear property, triangular property*

Counting the linear triples

To start with, let us enumerate by hand the values of $L(n)$ for a few small values of n.

$\mathbf{n}=1$: Since $S(1)$ has just the one triple $(1,1,1)$, and this does not have the linear property, $L(1)=0$.

$\mathbf{n}=2$: The triples in $S(2)$ which have the linear property are clearly $(1,1,2)$ with all its permutations; hence $L(2)=3$.

$\mathbf{n}=3$: The triples in $S(3)$ which have the linear property and have not been included in the previous list are $(1,2,3)$ with all its permutations; hence $L(3)=3+6=9$.

$\mathbf{n}=4$: The triples in $S(4)$ which have the linear property and have not been included in the previous list are $(1,3,4)$ and $(2,2,4)$ with all their permutations; hence $L(4)=9+6+3=18$.

$\mathbf{n}=5$: The triples in $S(5)$ which have the linear property and have not been included in the previous list are $(1,4,5)$ and $(2,3,5)$ with all their permutations; hence $L(5)=18+6+6=30$.

$\mathbf{n}=6$: The triples in $S(6)$ which have the linear property and have not been included in the previous list are $(1,5,6)$, $(2,4,6)$ and $(3,3,6)$ with all their permutations; hence $L(6)=30+6+6+3=45$.

Proceeding in this way, step by step, we construct by hand the following table of values of the L function:

n	1	2	3	4	5	6	7	8	9	10	11
$L(n)$	0	3	9	18	30	45	63	84	108	135	$\cdots$

Do you see any obvious pattern in the sequence of values of $L(n)$? A quick observation will reveal that all the L values are multiples of 3. This invites us to divide each number by 3; doing so, here is what we get:

$$0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45,\ \ldots.$$

Why, we have obtained the sequence of triangular numbers! What a nice surprise! We seem to have arrived at the following:

Conjecture 1. *$L(n)$ is equal to 3 times the $(n-1)^{\text{th}}$ triangular number, i.e.,*

$$L(n)=\frac{3n(n-1)}{2}.$$

Now we have:

$$\frac{3n(n-1)}{2}-\frac{3(n-1)(n-2)}{2}=3(n-1).$$

Therefore, Conjecture 1 is equivalent to the following:

Conjecture 2. *The number of positive integer triples whose largest number is n and which possess the linear property is $3(n-1)$.*

For example, take $n=4$. The positive integer triples with largest number 4 and which possess the linear property are $(1,3,4)$ and $(2,2,4)$ together with all their permutations; there are $6+3=9$ such triples. Note that $9=3\times(4-1)$.

Or take $n=5$. The positive integer triples with largest number 5 and which possess the linear property are $(1,4,5)$ and $(2,3,5)$ together with all their permutations; there are $6+6=12$ such triples. Note that $12=3\times(5-1)$.

Proof of Conjecture 2. Suppose that n is even. The positive integer triples with largest number n and which possess the linear property are the following:

$$\begin{array}{ll}
(1, n-1, n) & \text{together with its permutations,} \\
(2, n-2, n) & \text{together with its permutations,} \\
(3, n-3, n) & \text{together with its permutations,} \\
\dots & \dots \\
\left(\frac{1}{2}n-1, \frac{1}{2}n+1, n\right) & \text{together with its permutations,} \\
\left(\frac{1}{2}n, \frac{1}{2}n, n\right) & \text{together with its permutations.}
\end{array}$$

Except for the very last triple, all these triples have distinct elements. Therefore, the total number of permutations of all these triples is

$$6\left(\frac{1}{2}n-1\right)+3=3n-6+3=3(n-1).$$

Next, suppose that n is odd. The positive integer triples with largest number n and which possess the linear property are the following:

$$\begin{array}{ll}
(1, n-1, n) & \text{together with its permutations,} \\
(2, n-2, n) & \text{together with its permutations,} \\
(3, n-3, n) & \text{together with its permutations,} \\
\dots & \dots \\
\left(\frac{1}{2}(n-1), \frac{1}{2}(n+1), n\right) & \text{together with its permutations.}
\end{array}$$

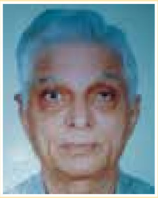
All these triples have distinct elements. Therefore, the total number of permutations of all these triples is

$$6\left(\frac{1}{2}(n-1)\right)=3(n-1).$$

The conjectured formula has been shown to hold in both the situations (when n is even and when n is odd). Hence it can be considered as proved. □

In Part II of the article, we shall study triangular triples.

Remark. The problem started here—that of counting positive integer triples possessing the linear property—seems ideal as exploration material for students of classes 8–10.



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