ClassRoom

# Theorem Concerning A Magic Triangle

SHAILESH A SHIRALI

Magic Triangles and Squares are often used as a 'fun activity' in the math class, but the magic of the mathematics behind such constructs is seldom explained and often left as an esoteric mystery for students. An article that can be used by teachers in the middle school (6-8) to justify to students that everything in mathematics has a reason and a solid explanation behind it. Plus a good way to practise some simple algebra.

According to the Wikipedia entry [2], "***A magic triangle*** *... is an arrangement of the integers from 1 to n on the sides of a triangle with the same number of integers on each side, ... so that the sum of integers on each side is a constant, the* ***magic sum*** *of the triangle.*" It then adds: *Unlike magic squares, there are different magic sums for magic triangles of the same order.* They are also known as ***perimeter magic triangles*** [1].

The number of integers on each side is called the ***order*** of the magic triangle. The order is clearly equal to $(n + 3)/3 = n/3 + 1$. (As a corollary, we see that $n$ must be a multiple of 3.) Figure 1 displays a third-order magic triangle with $n = 6$ and magic sum 9.

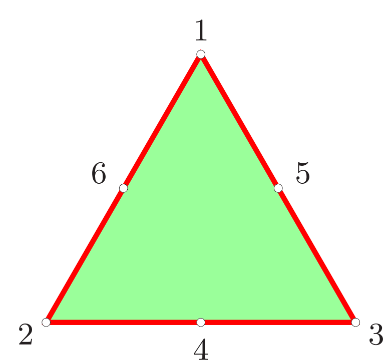


Figure 1. Third-order magic triangle with $n = 6$

*Keywords: Magic triangle, perimeter magic triangle, magic sum, arithmetic progression*

In this short note, we study magic triangles with $n = 9$ (which means that they are of order four). That is, we arrange the integers 1 to 9 on the sides of a triangle, with four integers on each side, in such a way that the sum of the integers on each side is the same. We discover, quite by chance, a striking result concerning the three numbers placed at the vertices. Specifically, we show the following:

**Theorem.** *The vertex numbers of a fourth-order magic triangle, when arranged in order, form an arithmetic progression.*

In fact, this is also true of third-order magic triangles, as Figure 1 illustrates. (We can see that the property holds for the magic triangle shown in the figure. But it is true for all third-order magic triangles. The proof of this is left as an exercise.)

Note that $x$, $y$, $z$ are sums of pairs of numbers.

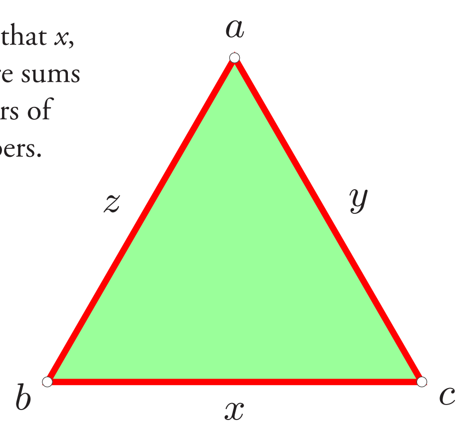


Figure 2. General relationships for fourth-order magic triangles ($n = 9$)

Proof. Let $a$, $b$, $c$ be the numbers at the three vertices (see Figure 2). Let $x$, $y$, $z$ be the sums of the other two numbers on the three edges, respectively ($x$ on edge $b$—$c$; $y$ on edge $c$—$a$; $z$ on edge $a$—$b$). Let $s$ be the magic sum of this triangle. Then we have the following relations:

$$\left.\begin{aligned} b + x + c &= s, \\ c + y + a &= s, \\ a + z + b &= s. \end{aligned}\right\} \qquad (1)$$

By adding the three relations we get:

$$2(a + b + c) + (x + y + z) = 3s. \qquad (2)$$

We also have:

$$a + b + c + x + y + z = 1 + 2 + 3 + \cdots + 9 = 45. \qquad (3)$$

Hence:

$$a + b + c = 3s - 45. \qquad (4)$$

So the sum of the numbers at the vertices is $3s - 45$. Note that this is a multiple of 3. So the sum of the vertex numbers is necessarily a multiple of 3.

Since $a + b + c \geq 1 + 2 + 3 = 6$ and $a + b + c \leq 9 + 8 + 7 = 24$, we get $6 \leq 3s - 45 \leq 24$, and therefore:

$$17 \leq s \leq 23. \qquad (5)$$

It follows that $s \in \{17, 18, 19, 20, 21, 22, 23\}$. We look at each possibility in turn.

**The case s = 17:** This possibility implies that $a + b + c = 6$ and can take place if and only if $\{a, b, c\} = \{1, 2, 3\}$. But in that case, the vertex numbers form an AP, as required. Figure 3 displays one of the magic triangles corresponding to this situation (there is one other such triangle which we will leave for the reader to find).

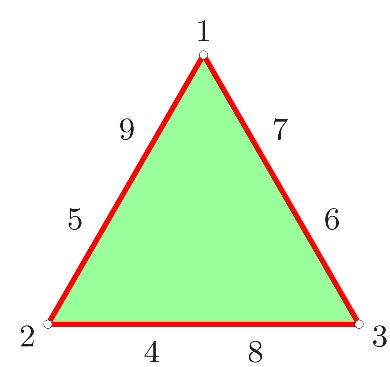


Figure 3. Fourth-order magic triangle with magic sum 17

**The case s = 18:** Rather to our surprise, we find that this possibility cannot occur at all! (So the assertion that the vertex numbers form an AP in this case is vacuously true, in the sense that it cannot be falsified.) But to see why takes a few steps which we now describe.

If $s$ = 18, then we must have $a + b + c = 9$. The only sets of three distinct integers whose sum is 9, the integers all lying between 1 and 9 (inclusive), are the following: {1, 2, 6}, {1, 3, 5} and {2, 3, 4}. (Please verify for yourself that these are the only possibilities.) In the latter two cases, the vertex numbers form APs; there is nothing more to show. So we focus on the first possibility, where the vertex numbers are 1, 2, 6. The situation is as depicted in Figure 4.

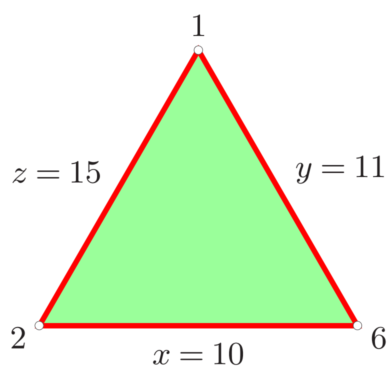


Figure 4. Analysis of a fourth-order magic triangle with magic sum 18

Consider the placement of the number 9. Can 9 be part of the pair whose sum is $x$? If so, then the other number of that pair must be 1. However, 1 has already been 'used up' (at a vertex). It follows that 9 cannot be part of the pair whose sum is $x$. Can 9 be part of the pair whose sum is $y$? If so, then the other number of that pair must be 2. However, 2 has already been 'used up' (at another vertex). It follows that 9 cannot be part of the pair whose sum is $y$. Can 9 be part of the pair whose sum is $z$? If so, then the other number of that pair must be 6. However, 6 has already been 'used up' (at yet another vertex). It follows that 9 cannot be part of the pair whose sum is $z$. All the possibilities have now been eliminated, which means that 9 has no place at all! But this means that the vertex numbers cannot be 1, 2, 6. Note the crucial role played by the number 9 in the above argument. Let us describe this role by saying that 9 *is a **witness** to showing the impossibility of having 1, 2, 6 as the vertex numbers.*

It turns out that the other two possibilities listed—with vertex numbers 1,3,5 and 2,3,4 respectively—also do not 'work'; in both cases, we are unable to construct the relevant magic triangle. And in both cases, the number 9 acts as a witness to show their impossibility. The relevant diagrams are shown in Figure 5. However, we omit the argument as it goes along exactly the same lines as the argument made above.

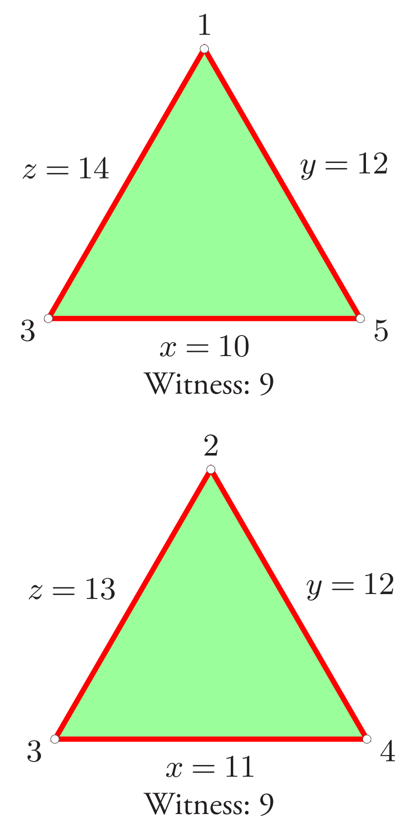


Figure 5. Analysis of fourth-order magic triangles with magic sum 18

It follows that if $s$ = 18, the statement that the vertex numbers form an AP is vacuously true.

**The case s = 19:** This possibility implies that $a + b + c = 12$. The only sets of three distinct integers, all between 1 and 9 (inclusive), who sum is 12, are the following: {1, 2, 9}, {1, 3, 8}, {1, 4, 7}, {1, 5, 6}, {2, 3, 7}, {2, 4, 6} and {3, 4, 5}. Of these, the ones that need closer examination are the following:

$$\{1, 2, 9\}, \{1, 3, 8\}, \{1, 5, 6\}, \{2, 3, 7\}. \qquad (6)$$

The first two cases are studied in Figure 6(a) and Figure 6 (b). In each case we need a witness that will play the role played by 9 in the earlier analysis. The relevant witnesses are listed alongside the captions. We leave it to the reader to verify that the witness plays its expected role in each case.

The remaining two cases are depicted in Figure 6 (c) and Figure 6 (d). As earlier, the relevant witnesses are listed alongside the captions. Once again, we leave the missing steps in the argument to be filled in by the reader. It follows that if $s = 19$, the vertex numbers form an AP in all the cases.

**The case s = 20:** This possibility implies that $a + b + c = 15$. We tackle the problem differently in this case. We must show that $a$, $b$, $c$ (in some order) form an AP. This is equivalent to showing that one of the numbers $a$, $b$, $c$ is 5. (The possible values for ($a$, $b$, $c$) then become (1, 5, 9), (2, 5, 8), (3, 5, 7), (4, 5, 6), all of which are APs.) Suppose not; that is, suppose that 5 occurs as an interior number on one of the edges, say on edge $b$—$c$. Let the remaining number (i.e., the fourth number) on edge $b$—$c$ be k. Then we have the following relation:

$$b + c + k + 5 = 20, \ \therefore b + c + k = 15. \qquad (7)$$

We also have $a + b + c = 15$. Comparing the two relations, we see that $a = k$. But this means that the same number has been used twice (it has acted as a witness, to use the former term). This is contrary to the stated requirement that no number can be used more than once. It follows

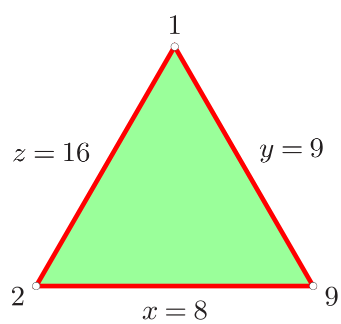


Figure 6 (a). Witness: 7

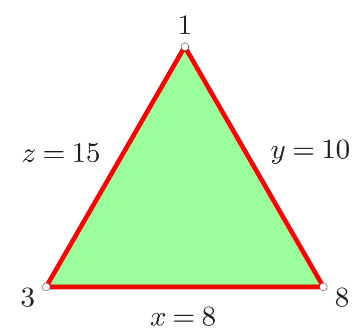


Figure 6 (b). Witness: 7

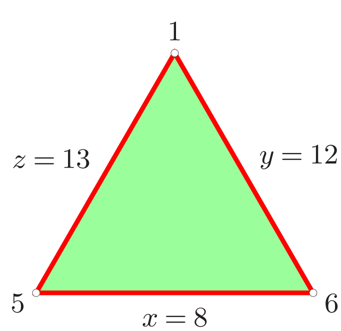


Figure 6 (c). Witness: 7

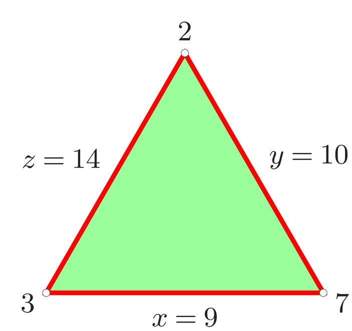


Figure 6 (d). Witness: 7

Figure 6. Analysis of fourth-order magic triangles with magic sum 19

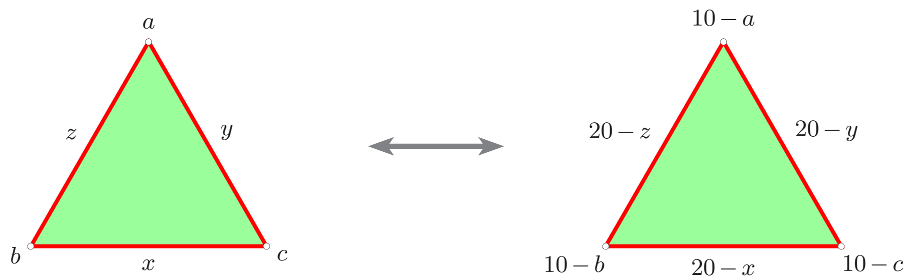


Figure 7. Fourth-order magic triangles with magic sums s and 40–s

that one of the numbers $a$, $b$, $c$ is 5, and therefore that $a$, $b$, $c$ (in some order) form an AP.

**The cases s = 21, 22, 23:** The remaining cases ($s$ = 21, 22, 23) are best handled by appealing to symmetry. Given a fourth-order magic triangle with magic sum $s$, if we replace every entry by its tens-complement, i.e., we replace $a$, $b$, $c$, . . . by $10 - a$, $10 - b$, $10 - c$, . . ., respectively (this means that we replace $x$, $y$, $z$ by $20 - x$, $20 - y$, $20 - z$, respectively), we get a fourth-order magic triangle whose magic sum is $40 - s$ (see Figure 7).

If the magic sum $s$ is one of the numbers 21, 22, 23, then $40 - s$ is one of the numbers 19, 18, 17, which means that the earlier analysis applies. Since the vertex numbers for the modified magic triangles do form an AP (proved above), the same must be true for the vertex numbers of the original magic triangles.

We remark in closing that the nonexistence of a magic triangle with $s = 18$ implies, in the light of the above remark, the nonexistence of a magic triangle with $s = 22$. Here too, the assertion that "the vertex numbers form an AP" is vacuously true.

## References


1. Harvey Heinz. "Perimeter Magic Triangles." http://www.magic-squares.net/perimeter.htm#Order4PerimeterMagicTriangles
2. Wikipedia. "Magic triangle (mathematics)." https://en.wikipedia.org/wiki/Magic_triangle_(mathematics)



**SHAILESH SHIRALI** is Director and Principal of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.