feature

Fujimoto's Approximation Method

How do you Divide a Strip into Equal Fifths?

How can a simple series of folds on a strip of paper be a mathematical exercise? The article describes not only the how but also the why.....

SHIV GAUR

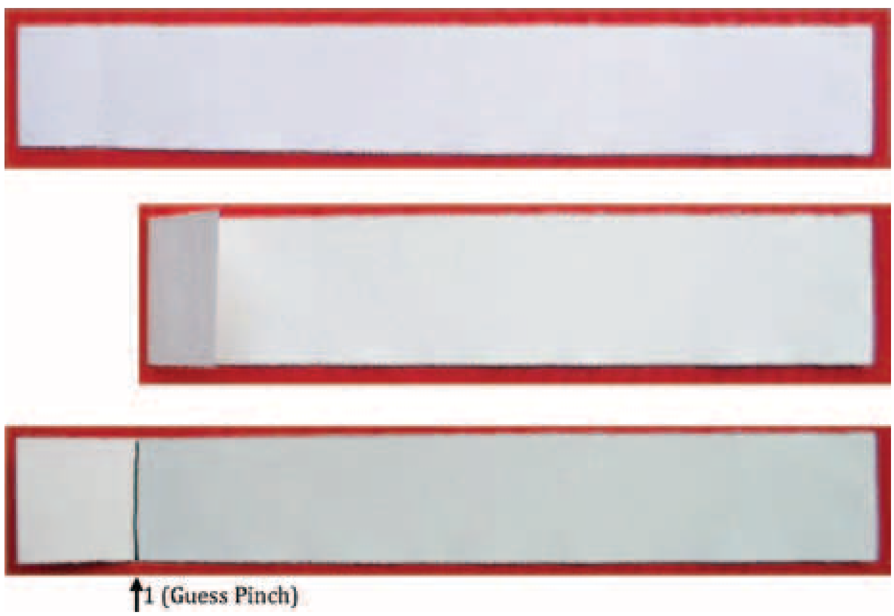
A key principle for learning mathematics is to move from the concrete to the abstract, which implies having "active mathematical experiences" first. Paper folding is one avenue for such experiences. Concepts of perpendicularity, parallelism, similarity, congruence and symmetry are easily experienced through paper folding activities and provide an experiential base for further learning. Paper folding also lends itself readily to explorations, visual proofs and constructions. Angle trisection and doubling of a cube which are not possible with straight edge and compass and the traditional rules of Euclidean constructions are possible using paper folding.

Origami (Japanese-ori: to fold; kami: paper) is the art of paper folding. By a sequence of folds, a flat piece of paper is turned into an animal, flower or a box. In addition to conventional folding (Flat Origami), the art encompasses such genres as Modular Origami (many identical units combined to form decorative polyhedra), and Composite Origami (objects folded from two or more sheets of paper). Origami also provides a highly engaging and motivating environment within which children extend their geometric experiences and powers of spatial visualization.

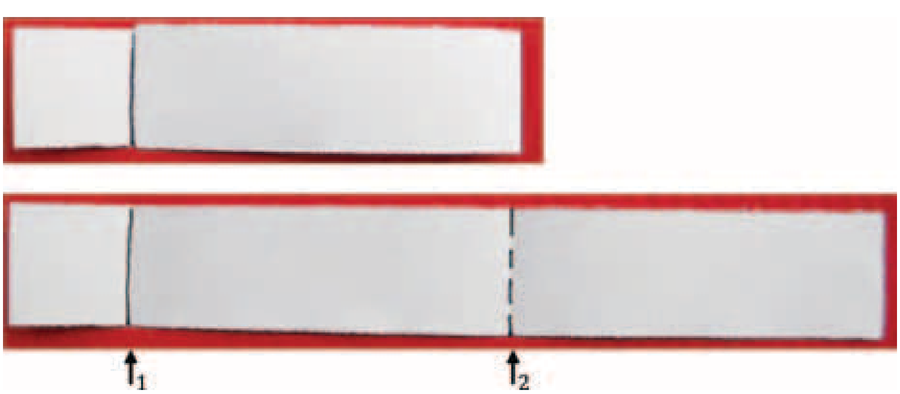
In 1893 T. Sundara Row published his book "Geometric Exercises in Paper Folding" which is considered a classic and still in print. In the development of Origami over the years a number of ideas and techniques have emerged which have mathematical underpinnings, such as Haga's theorems, the Huzita-Hatori Axioms and Fujimoto's Approximation Method to name a few.

In origami it is very common to fold the side of a square piece of paper into an equal number of parts. If the instructions for a particular model ask for it to be folded in half or into quarters or eighths, then it's easy to do so. The difficulty arises if they ask for an equal fifths or any equal 'odd' number of folds. Thankfully there is an elegant and popular method called **Fujimoto's Approximation Method**. Here are the steps for dividing a strip of paper into equal fifths. The photographs are included to show the steps more clearly.

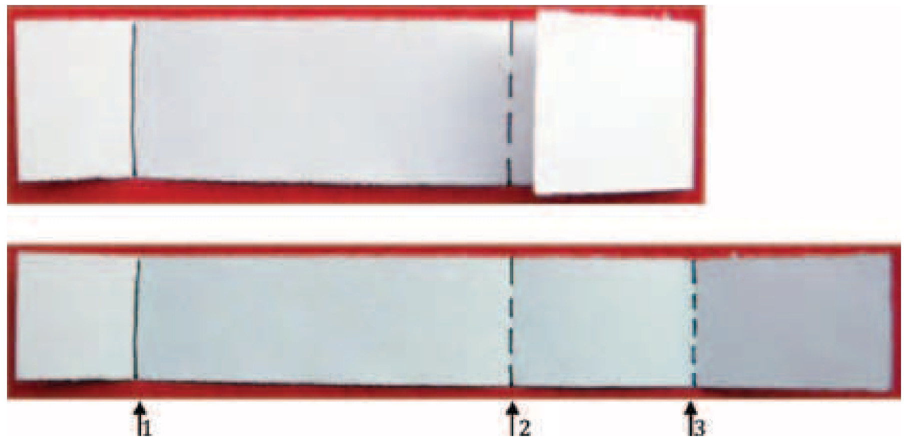
Step 01: Make a **guess pinch** where you may think $\frac{1}{5}$ might be, say on the *left side* of the paper.



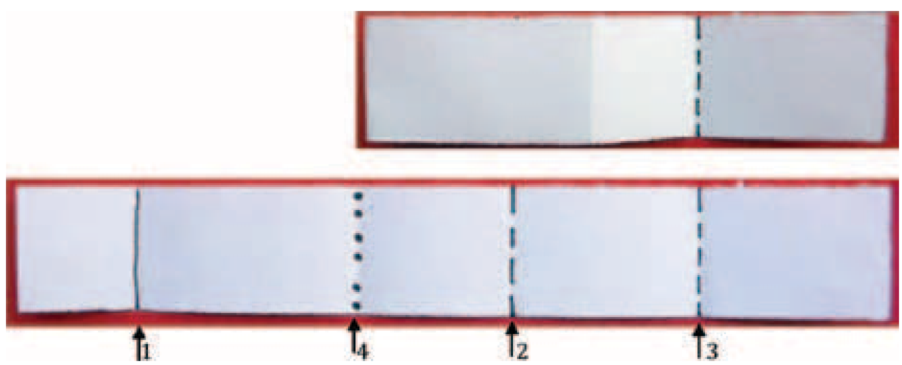
Step 02: To the right side of this guess pinch is approximately $\frac{4}{5}$ of the paper. Pinch this side **in half**.



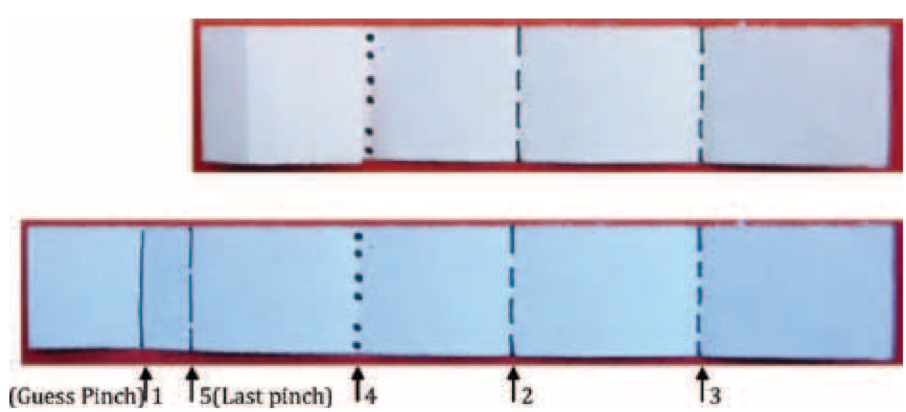
Step 03: The last pinch is near the $\frac{3}{5}$ mark from the left side. To the right side of this is approximately $\frac{2}{5}$ of the paper. Pinch this side **in half**.



Step 04: Now you have a $\frac{1}{5}$ mark on the right. To the left of this is approximately $\frac{4}{5}$. Pinch this side **in half**.



Step 05: This gives a pinch (the dotted line) close to the $\frac{2}{5}$ mark from the left. Pinch the left side of this **in half**. This last pinch will be **very close** to the actual $\frac{1}{5}$ mark!



The set of 3 long dashed lines is the **last pinch mark** very close to the **guess pinch** and a better approximation of $\frac{1}{5}$ than the original guess pinch.

Iteration of the last four steps starting with the **last pinch** as the **new guess mark** helps in finding the fifth mark. The closer the "guess pinch" is to the actual fifth, the fewer the number of iterations.

Why does this work?

Naturally, the question arises: *Why does this work?* Regarding the strip as 1 unit in length, the initial "guess pinch" can be thought of as being at distance

$$\frac{1}{5} + e$$

from the left side, where e represents the initial error. (This could be positive or negative, depending on which side we have erred.) Now with each subsequent fold the error gets halved!

For, in steps 2, 3, 4 and 5, we find that the distances of the latest pinch folds from the left side are, respectively (please verify this):

$$\frac{3}{5} - \frac{e}{2}, \quad \frac{4}{5} + \frac{e}{4}, \quad \frac{2}{5} - \frac{e}{8}, \quad \frac{1}{5} + \frac{e}{16}.$$

The sign of the error alternates between plus and minus. The crucial part is that the last error is $\frac{1}{16}$ of the original one! So each round of this procedure brings down the error by a factor of 16.

Observe that we have traversed pinch marks which cover all the multiples of $\frac{1}{5}$.

Another question is, what would be the procedure for folding a paper into n equal parts, where n is a given odd number?

The general idea in the Fujimoto algorithm is to make an approximate $\frac{1}{n}$ pinch, say from the left hand side. The crease line can be viewed as a fraction of the paper, either from the left side or the right side. Since n is odd, just one of the two fractions will have an even numerator. To get the next crease line, we fold in half that part of the strip from that edge of the paper which corresponds to the *even* numerator, to the latest crease line. Eventually we will reach a pinch mark which provides a new, more accurate approximation for $\frac{1}{n}$ of the paper, since the error gets reduced by half each time the paper is folded in half.

Now try on your own to get a similar method for folding into equal thirds.

Fujimoto's method provides an insight into why clumsy origami folders manage to do a fairly good job of models with intricate folds!

Appendix:
Who is Shuzo Fujimoto?

Fujimoto described this approximation method in a book written in Japanese and published in 1982 (S. Fujimoto and M. Nishiwaki, *Sojo Suru Origami Asobi Eno Shotai* ("Invitation to Creative Origami Playing"), Asahi Culture Center, 1982). Read what has been written about Fujimoto at this link: http://www.britishorigami.info/academic/lister/tessel_begin.php.

A B.Ed. and MBA degree holder, SHIV GAUR worked in the corporate sector for 5 years and then took up teaching at the Sahyadri School (KFI). He has been teaching Math for 12 years, and is currently teaching the IGCSE and IB Math curriculum at Pathways World School, Aravali (Gurgaon). He is deeply interested in the use of technology (Dynamic Geometry Software, Computer Algebra System) for teaching Math. His article "Origami and Mathematics" was published in the book "Ideas for the Classroom" in 2007 by East West Books (Madras) Pvt. Ltd. He was an invited guest speaker at IIT Bombay for TIME 2009. Shiv is an amateur magician and a modular origami enthusiast. He may be contacted at shivgaur@gmail.com