feature

## Many Ways to QED

# The Pythagorean Theorem

### Taking note of a collective of contributors

*How do I prove thee? Can I count the ways? A look at the wide variety of methods used to prove the theorem of Pythagoras.*

CoMaC

Not only is the theorem of Pythagoras ('PT' for short) the best known mathematical theorem of any kind, it also has the record of having been proved in a greater number of ways than any other result in mathematics, and by a huge margin: it has been proved in more than three hundred and fifty different ways! (So the relation between the PT and the rest is a bit like the relation between Sachin Tendulkar and the rest ….) There is more: though by name it is inextricably linked to one particular individual (Pythagoras of ancient Greece), as a geometric fact it was *independently* known in many different cultures. (See the article by J Shashidhar, elsewhere in this issue, for more on the history of the PT.) We do not know whether they proved the theorem and if so how they did it, but they certainly knew it was true!

In this article we describe a few proofs of this great and important theorem.

## Statement of the PT

Here is how Euclid states it: *In a right triangle, the square on the hypotenuse equals the sum of the squares on the two legs of the triangle*. ('Right triangle' is a short form for 'right-angled triangle'. The 'legs' of a right triangle are the two sides other than the hypotenuse.) Thus, in Figure 1(i) where $\measuredangle A$ is a right angle, we have:

Square *BHIC* = Square *ADEB* + Square *ACFG*.

The way we state the theorem nowadays is: *In a right triangle, the square* ***of*** *the hypotenuse equals the sum of the squares* ***of*** *the two legs of the triangle*. Note the change: *'on' has been replaced by 'of'*. Therefore: *In Figure 1(ii)*, $a^2=b^2+c^2$. This is not merely a change of language. In Euclid's version, it is a statement about areas; in the latter one, it is a statement about lengths. Of course the two versions are equivalent to one another (thanks to the formula for area of a square), and both offer opportunities for generalization; but the second one tells us something about the structure of the space in which we live. Today, this is the preferred version.

## Euclid's proof

This has been sketched in Figure 2. The description given alongside gives the necessary steps, and we shall not add anything further here. Note that the reasoning is essentially geometrical, using congruence theorems. *No algebra is used*.

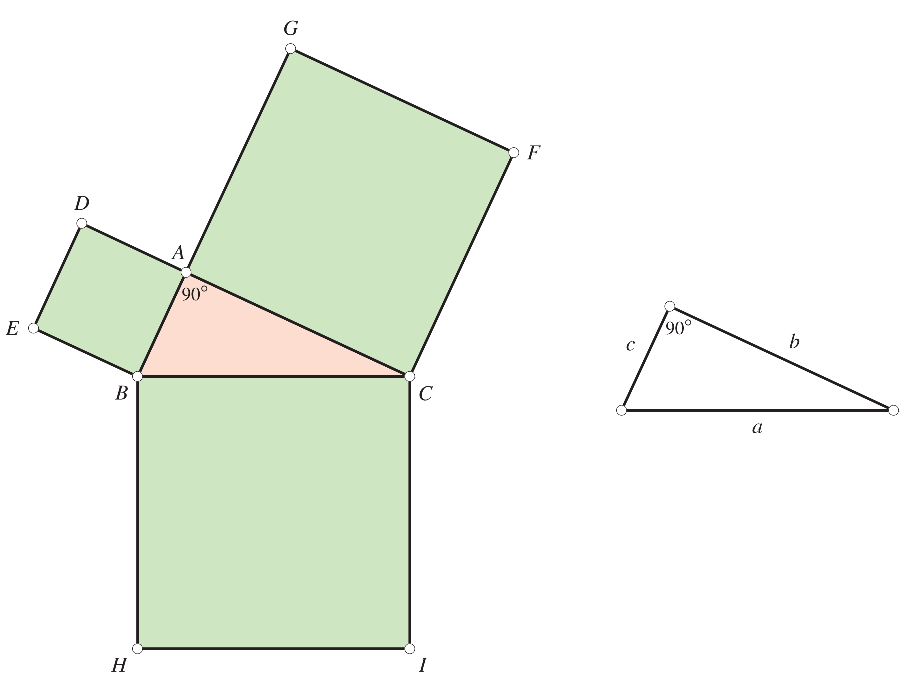
## Bhaskara II's proof

The proof given by Bhaskara II, who lived in the 12th century in Ujjain, is essentially the same as the one described in the origami article by V S S Sastry elsewhere in this issue; but the triangles are stacked differently, as shown in Figure 3. Let a right $\triangle ABC$ be given, with $\measuredangle A=90°$; let sides *BC*, *AC*, *AB* have lengths *a*, *b*, *c*.

The argument given in Figure 3 shows that $(b+c)^2=4\ \left(\frac{1}{2}bc\right)+a^2$, and hence that $a^2=b^2+c^2$. Note that all we have done is to 'keep accounts': that is, account for the total area in two different ways.

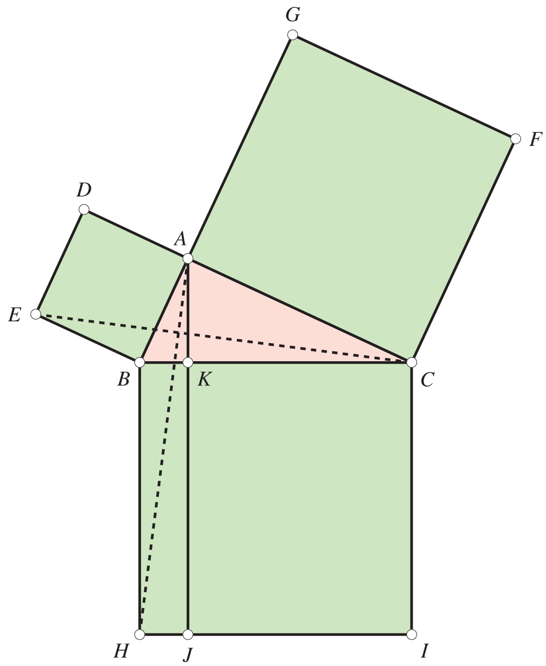
Bhaskara's proof is a beautiful example of a 'proof without words'. The phrase 'without words' is not be taken too literally; words are certainly used, but kept to a minimum. We shall see many more examples of such visual proofs in future issues of this magazine.

Properly speaking, we must justify certain claims we have made in this proof (and this, typically, is the case for all proofs-without-words); for



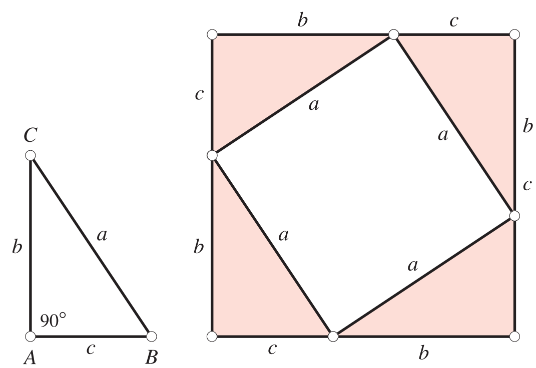

(i) (ii)

Fig 1:
Two contrasting versions of Pythagoras's theorem.
In (i), Square BHIC =Square ACFG + Square ADEB;
In (ii), $a^2 = b^2 + c^2$




- Draw $AK \perp BC$ and extend $AK$ to meet $HI$ at $J$.
- **Claim.** Square $ADEB$ and rectangle $BHJK$ have equal area. (Proof: Given below.)
- Draw segments $AH$ and $EC$.
- $\triangle ABH \cong \triangle EBC$ ('SAS' congruence)
- Area of $\triangle ABH$ is half the area of rectangle $BHJK$, and area of $\triangle EBC$ is half the area of square $ADEB$.
- Hence square $ADEB$ and rectangle $BHJK$ have equal area.
- Similarly, square $ACFG$ and rectangle $KJIC$ have equal area.
- By addition, the PT follows.

Fig. 2 Euclid's proof, which uses standard geometrical results on congruence




Take 4 identical copies of $\triangle ABC$ and arrange them as shown. They make up a square with side $b+c$, but with a square 'hole' in the middle, with side $a$. The area of each right triangle is $\frac{1}{2}bc$, the area of the hole is $a^2$, and the area of the large square is $(b+c)^2$.

Fig. 3 The proof by Bhaskara II. See reference (1) for details

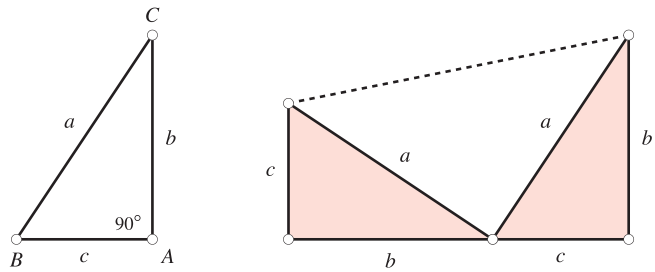
example: (i) why the 'hole' is a square with side $a$, (ii) why the entire figure is a square with side $b+c$. For this we must show that angles which 'look like right angles' are indeed right angles, and angles which 'look like straight angles' are indeed straight angles. But these justifications are easily given — please do this on your own.

According to legend, Bhaskara did not offer any explanations (we presume therefore that he agreed with the philosophy of a proof without words); he simply drew the diagram and said "Behold! " — assuming no doubt that the reader would be astute enough to work out the details mentally, after gazing for a while at the diagram!

## Garfield's proof

The proof given by James Garfield in 1876 is of a similar nature. Garfield was a US Senator at the time he found the proof, and later (1881) became President of the USA. Unfortunately he fell to an assassin's bullet later that same year, and died a slow and painful death.

Garfield's argument is sketched in Figure 4. The trapezium has parallel sides of lengths $b$ and $c$, and the perpendicular distance between them is $b+c$; its area is therefore $\frac{1}{2}(b+c)^2$. The two right triangles have area $\frac{1}{2}bc$ each. Hence we have: $\frac{1}{2}(b+c)^2 = bc + \frac{1}{2}a^2$. On simplifying this we get $a^2 = b^2 + c^2$.




Take two copies of $\triangle ABC$ and arrange them as shown. On drawing the dashed line, we get a trapezium made up of two right triangles and half a square of side $a$.

Fig. 4 The proof by Senator Garfield. See reference (2) for details.

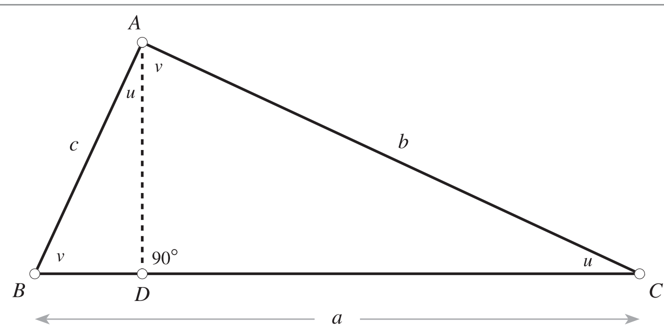



Fig. 5

## A proof based on similarity

Next, we have a proof based on similarity of triangles, also given by Euclid in his text ELEMENTS.

Figure 5 depicts the right triangle $ABC$ in which $\measuredangle A = 90°$, with a perpendicular $AD$ drawn from $A$ to the base $BC$. The two angles marked $u$ are equal, as are the two angles marked $v$. So we have the similarities $\triangle ABC \sim \triangle DBA \sim \triangle DAC$, and we deduce that

$$\frac{BD}{BA} = \frac{AB}{CB} = \frac{c}{a}, \qquad \frac{DC}{AC} = \frac{AC}{BC} = \frac{b}{a}.$$

These imply that

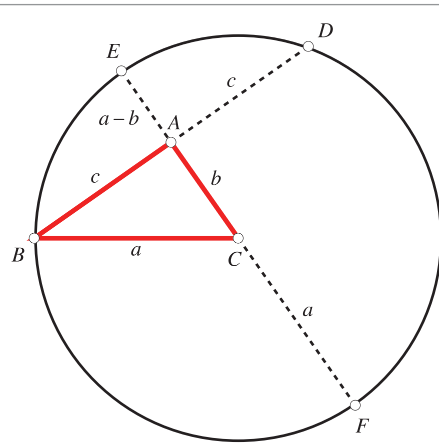
$$BD = c \times \frac{c}{a} = \frac{c^2}{a}, \quad DC = b \times \frac{b}{a} = \frac{b^2}{a}.$$

Since $BD + DC = a$, we get

$$\frac{c^2}{a} + \frac{b^2}{a} = a,$$

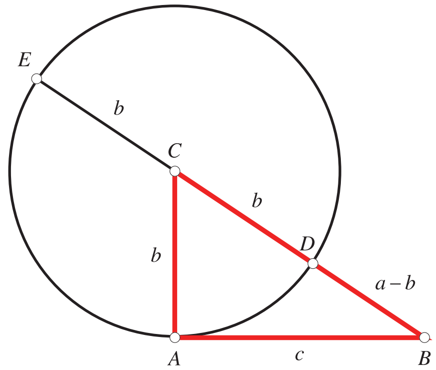
so $a^2 = b^2 + c^2$.

Observe that this proof yields some additional relations of interest; for example, $AD^2 = BD \quad DC$, and $BD : DC = AB^2 : AC^2$.




- Given: $\triangle ABC$, with $\measuredangle A = 90°$
- Circle is drawn with centre $C$, radius $a$; it passes through $B$
- $BA$ is extended to $D$, and segment $AC$ to $E$ and to $F$
- Then $CF = a$, $AE = a - b$
- Next, $AD = c$ (for $CA \perp BD$, so $A$ must be the midpoint of chord $BD$)
- Now apply chord theorem to chords $BD$, $EF$: we get $c^2 = (a+b)(a-b)$, and hence $a^2 = b^2 + c^2$

Fig. 6 Proof #63 from the compilation by E S Loomis (reference 3)




- Given $\triangle ABC$, with $\measuredangle A = 90°$
- Draw circle with centre $C$, passing through $A$. The radius of the circle is $b$.
- Let the circle cut ray $BC$ at points $D, E$.
- $BA = c, \quad BC = a$
- $BD = a - b, \quad BE = a + b$
- $BA^2 = BD \times BE$ by chord theorem
- Hence $c^2 = (a+b)(a-b)$. Expanding we get $a^2 = b^2 + c^2$

Fig. 7 Another proof based on the intersecting chords theorem

## A proof based on the intersecting chords theorem

Next, we have a lovely proof based on the intersecting chord theorem ("If $UV$ and $LM$ are two chords of a circle, intersecting at a point $T$, then $UT \quad VT = LT \quad MT$"), which is a well known result in circle geometry (and, importantly, its proof does not depend on the PT). The construction and proof are fully described in Figure 6.

## Another proof based on the intersecting chords theorem

A corollary of the intersecting chords theorem is the following: "If from a point $P$ outside a circle, a tangent $PT$ is drawn and also a secant $PQR$, cutting the circle at $Q$ and $R$, then $PQ \quad PR = PT^2$." We may use this to get yet another proof. The details have been given in Figure 7.

## Are these proofs really different from one another?

Yes, indeed! Euclid's proof is about the geometric notion of area; it uses standard theorems of congruence, and does not require any algebraic ideas whatever. Bhaskara's proof too uses the notion of area, but requires: (i) the fact that the area of a rectangle with sides $x$ and $y$ is $xy$ (and therefore that the area of a right triangle with legs $x$ and $y$ is $\frac{1}{2}xy$, and the area of a square of side $s$ is $s^2$); (ii) the formula for the expansion of $(b+c)^2$. Likewise for Garfield's proof. Finally, the proof by similarity and the two proofs based on the intersecting chords theorem have nothing to do with area at all! — they deal with *lengths*, and it is purely by algebraic manipulations that the relation $a^2 = b^2 + c^2$ emerges.

### References


1. http://www.robertnowlan.com/pdfs/Bhaskara.pdf
2. http://nrich.maths.org/805 (for the proof by Senator Garfield)
3. Elisha S Loomis, *The Pythagorean Proposition*, NCTM, USA
4. http://www.cut-the-knot.org/pythagoras/index.shtml


Reference (3) contains no less than 371 proofs of the PT, and 96 of these are given in reference (4)!