feature

Heron's Formula for Area of a Triangle

One Formula - Two Derivations

Cleverly used algebra in an old familiar formula for the area of a triangle in terms of its base and height enables the formula to be restated in terms of the sides of the triangle. An account of the derivation.

C⊗MαC

With what relief a student uses the simple formula 'Area of a triangle = $\frac{1}{2}$base × height'! Though Heron's formula for the area in terms of its three sides has a pleasing symmetry convenient for memorization, it often seems cumbersome in comparison. A look at how this formula is derived will perhaps enable the student to remember and appreciate the formula, not for this reason but for the sheer elegance of the derivation.

The formula is well known: if the sides of the triangle are a, b, c, and its semi-perimeter is $s = \frac{1}{2}(a+b+c)$, then its area Δ is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\,.$$

We present two proofs of the theorem. The first one is a consequence of the theorem of Pythagoras, with lots of algebra thrown in. It is striking to see how heavily the humble 'difference of two squares' factorization formula is used.

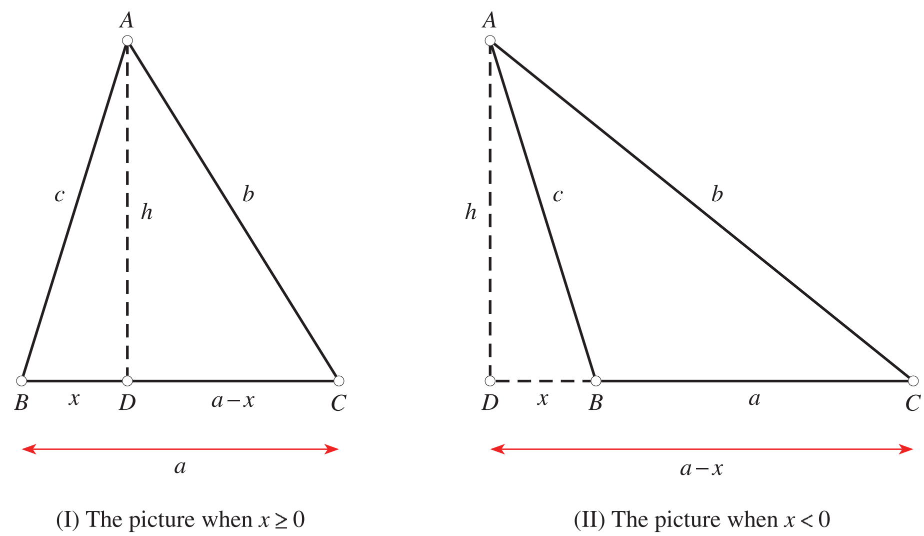


Fig 1: The figure as it looks when $\measuredangle B$ is acute, and when it is obtuse

Proof based on the theorem of Pythagoras

The proof has been described with reference to Figure 1 (I) and Figure 1 (II). Given the sides a, b, c of $\triangle ABC$, let the altitude AD have length h. The area of $\triangle ABC$ is $\frac{1}{2}ah$. To find h in terms of a, b, c we use Pythagoras's theorem. Let $BD = x$, $DC = a - x$. (The notation does not imply that x must lie between 0 and a. Indeed, if $\measuredangle B$ is obtuse then $x < 0$, and if $\measuredangle C$ is obtuse then $x > a$. See Figure 1 (II). Please draw your own figure for the case when $x > a$.) Then:

$$h^2 + x^2 = c^2, \quad h^2 + (a-x)^2 = b^2.$$

Subtract the second equation from the first one:

$$2ax - a^2 = c^2 - b^2, \quad \therefore\ x = \frac{c^2 + a^2 - b^2}{2a}.$$

Since $h^2 + x^2 = c^2$, this yields:

$$\begin{aligned} h^2 &= c^2 - \left(\frac{c^2 + a^2 - b^2}{2a}\right)^2 \\ &= \left(c - \frac{c^2 + a^2 - b^2}{2a}\right) \times \left(c + \frac{c^2 + a^2 - b^2}{2a}\right) \\ &= \frac{2ac - c^2 - a^2 + b^2}{2a} \times \frac{2ac + c^2 + a^2 - b^2}{2a}, \end{aligned}$$

$$\therefore\ 4a^2h^2 = (2ac - c^2 - a^2 + b^2) \times (2ac + c^2 + a^2 - b^2).$$

The area of the triangle is $\Delta = \frac{1}{2}ah$, so $16\Delta^2 = 4a^2h^2$, i.e. :

$$16\Delta^2 = (2ac - c^2 - a^2 + b^2) \times (2ac + c^2 + a^2 - b^2).$$

$$\begin{aligned} &= [b^2 - (c-a)^2] \times [(c+a)^2 - b^2] \\ &= (b - c + a)(b + c - a)(c + a + b)(c + a - b) \\ &= (2s - 2c)(2s - 2a)(2s)(2s - 2b), \end{aligned}$$

$$\therefore \Delta^2 = s(s-a)(s-b)(s-c),$$

therefore $\Delta = \sqrt{(s-a)(s-b)(s-c)}$.

Another proof

We now present an entirely different proof. It is based on a note written by R Nelsen (see reference (1)) and uses two well known results:

- *If* α, β, γ *are three acute angles with a sum of* 90°, *then*

(1) $\tan\alpha \tan\beta + \tan\beta \tan\gamma + \tan\gamma \tan\alpha = 1.$

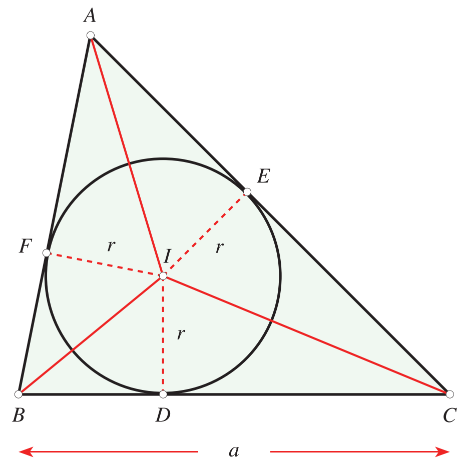
Nelsen gives a 'proof without words' but we simply use the well known addition formula for the tangent function. Since $\alpha + \beta$ and γ have a sum of 90° their tangents are reciprocals of one another:

$$\tan(\alpha + \beta) = \frac{1}{\tan\gamma}$$

But we also have:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\therefore \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{1}{\tan\gamma}$$



The three radii of the incircle (shown dashed) are also altitudes of $\triangle IBC$, $\triangle ICA$ and $\triangle IAB$. The bases of these triangles are a, b and c, so their areas are $\frac{1}{2}ar$, $\frac{1}{2}br$ and $\frac{1}{2}cr$.

Hence $\Delta = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$. Factoring this we get: $\Delta = \frac{1}{2}(a+b+c)r = sr$.

Fig 2: Proof of the area formula $\Delta = rs$

Cross-multiplying and transposing terms, we get (1).

- *If s is the semi-perimeter of a triangle, and r is the radius of its incircle, then its area* Δ *is given by* $\Delta = rs$*.* The proof is given in Figure 2; it is almost a proof without words!

Now we move to Figure 3 which is the same as Figure 2 but with some extra labels. The two lengths marked x are equal ("The tangents from a point outside a circle to the circle have equal length"), as are the two lengths marked y, and the two lengths marked z; and also the two angles marked α, the two angles marked β, and the two angles marked γ.

Consider the angles marked α, β, γ:

$$\alpha = \measuredangle FAI = \measuredangle EAI,$$

$$\beta = \measuredangle DBI = \measuredangle FBI,$$

$$\gamma = \measuredangle DCI = \measuredangle ECI.$$

Since $\alpha + \beta + \gamma = 90°$, by (1) we have:

$$\tan\alpha \tan\beta + \tan\beta \tan\gamma + \tan\gamma \tan\alpha = 1.$$

But from Figure 3,

$$\tan\alpha = \frac{r}{x}, \quad \tan\beta = \frac{r}{y}, \quad \tan\gamma = \frac{r}{z}.$$

Therefore we get, by substitution,

$$\text{(2)} \qquad \frac{r^2}{xy} + \frac{r^2}{yz} + \frac{r^2}{zx} = 1, \quad \therefore \frac{r^2(x+y+z)}{xyz} = 1.$$

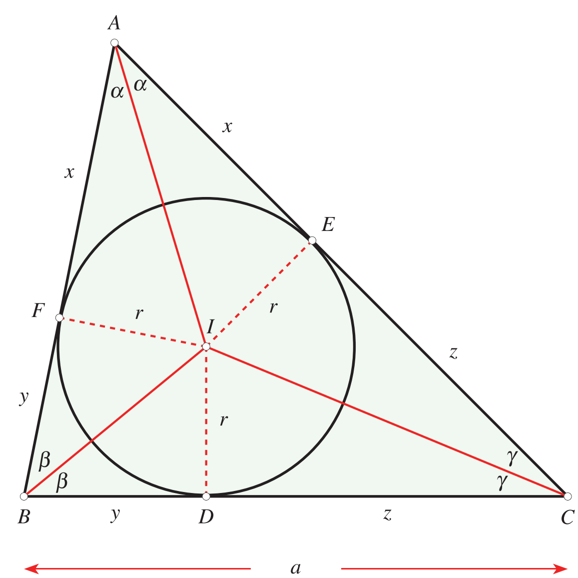


Fig 3

Now $x+y+z = s$ (the semi-perimeter); and since $y+z = a$, we have $x = s-a$. In the same way, $y = s-b$ and $z = s-c$. So result (2) may be rewritten as:

$$\frac{r^2 s}{(s-a)(s-b)(s-c)} = 1,$$

$$\text{i.e.,} \quad \frac{r^2 s^2}{s(s-a)(s-b)(s-c)} = 1.$$

Since $rs = \Delta$ this yields:

$$\Delta^2 = s(s-a)(s-b)(s-c),$$

and we have obtained Heron's formula.

Who was Heron?

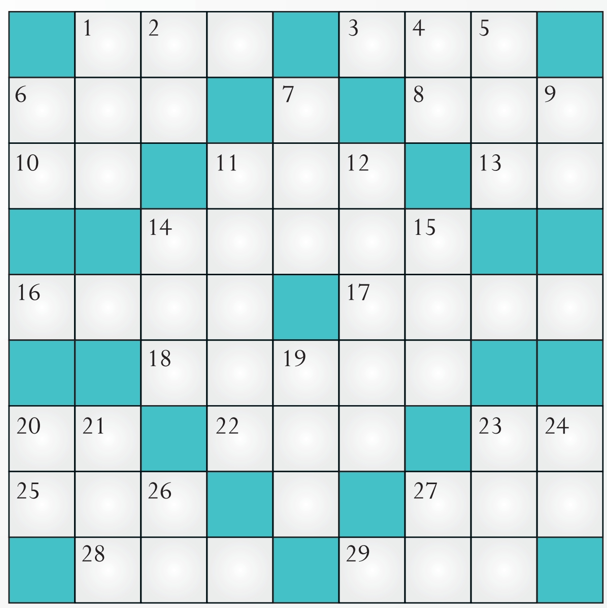
Heron lived in the first century AD, in the Roman town of Alexandria of ancient Egypt. He was a remarkably inventive person, and is credited with inventing (among other things) a wind powered machine and a coin operated vending machine — perhaps the first ever of its kind! For more information please see reference (2).

References

1. Roger B. Nelsen, *Heron's Formula via Proofs without Words*. College Mathematics Journal, September, 2001, pages 290–292
2. http://en.wikipedia.org/wiki/Hero_of_Alexandria.

number crossword-1

by D.D. Karopady



Clues Across

1: 29A minus first two digits of 14A
3: 16A minus 102
6: One less than sum of internal angles of a triangle
8: Three consecutive digits in reverse order
10: Sum of the digits of 14A reversed
11: Even numbers in a sequence
13: A number usually associated with π
14: The product of 29A and 25A
16: Sum of internal angles of an Octagon
17: The product of 6D and 28A
18: Palindrome with 4,3
20: The product of 27D and 8
22: 2 to the power of the sum of the digits of 6D
23: 5 more than 13A
25: Consecutive digits
27: 6D reduced by 1 and then multiplied by 10
28: Sum of two angles of an equilateral triangle
29: A perfect cube

Clues Down

1: 20A reduced by 2 and then multiplied by 2
2: The largest two digit number
4: Four times the first two digits of 1A
5: 20D times 10 plus 9D
6: Hypotenuse of right angle triangle with sides 8 and 15
7: 1A minus 25
9: 26D plus 10
11: First two digits are double of last two digits in reverse
12: 11D plus 18A minus 3230
14: The middle digit is the product of the first and the last digit
15: 27A times 5 then add 4
19: Two times 29A plus 26D in reverse
20: Largest two digit perfect square
21: Half of 22A written in reverse
23: 29A plus half a century
24: Difference between last 2 digits & first 2 digits of 16A
26: Four squared times two
27: A prime number