

The Most Famous Theorem

# The Theorem of Pythagoras

feature

## Understanding History – Who, When and Where

*Does mathematics have a history? In this article the author studies the tangled and multi-layered past of a famous result through the lens of modern thinking, looking at contributions from schools of learning across the world, and connecting the mathematics recorded in archaeological finds with that taught in the classroom.*

SHASHIDHAR JAGADEESHAN

Imagine, in our modern era, a very important theorem being attributed to a cult figure, a new age guru, who has collected a band of followers sworn to secrecy. The worldview of this cult includes number mysticism, vegetarianism and the transmigration of souls! One of the main preoccupations of the group is mathematics: however, all new discoveries are ascribed to the guru, and these new results are not to be shared with anyone outside the group. Moreover, they celebrate the discovery of a new result by sacrificing a hundred oxen! I wonder what would be the current scientific community's reaction to such events.

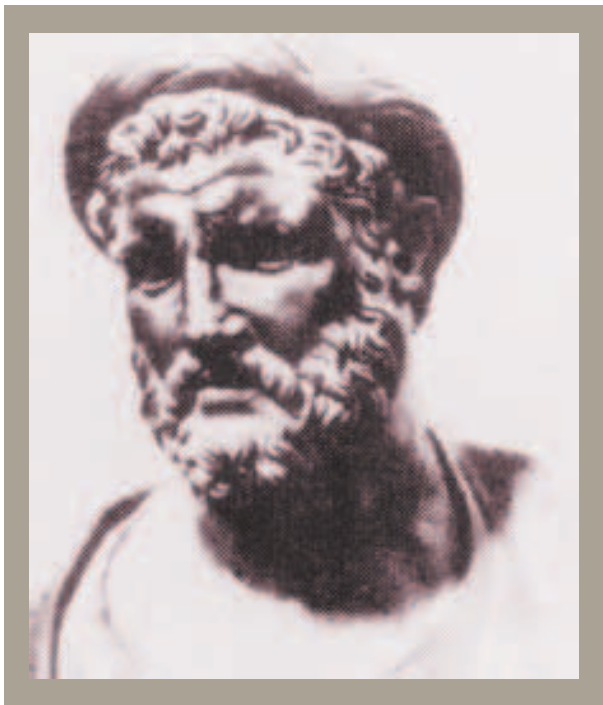
This is the legacy associated with the most 'famous' theorem of all times, the Pythagoras Theorem. In this article, we will go into the history of the theorem, explain difficulties historians have with dating and authorship and also speculate as to what might have led to the general statement and proof of the theorem.

## Making sense of the history

Often in the history of ideas, especially when there has been a discovery which has had a significant influence on mankind, there is this struggle to find out who discovered it first. This search is very often coloured by various biases and obscured by the lack of authentic information and scholarship. The Pythagoras Theorem suffers from the same fate. In this article I hope to give a summary of current understanding and, not being an expert historian, would like to state right at the beginning that I might have left out some major contribution.

Before proceeding further let us, at the cost of redundancy, recall the Pythagoras Theorem as stated by Euclid (I.47 of the *Elements*) and refer to it from now on as PT (P here can stand for Pythagoras or 'Preeminent!').

Obviously the first challenge for historians is the name.



Pythagoras (Approx 572 BC to 475 BC)

Why Pythagoras? Greek scholars seem to be in agreement that the first person to clearly state PT in all its generality, and attempt to establish its truth by the use of rigorous logic (what we now call mathematical proof), was perhaps Pythagoras

of Samos. We actually know very little about Pythagoras, and what we do know was written by historians centuries after he died.

Legend has it that Pythagoras was born around 572 B.C. on the island of Samos on the Aegean Sea. He was perhaps a student of Thales, a famous Greek philosopher and mathematician who was born half a century before Pythagoras. It is believed that Pythagoras travelled to Egypt, Babylon and even to India before he

returned to Croton, a Greek settlement in south-east Italy. Here he seems to have gathered a group of followers forming what we call the Pythagorean sect, with beliefs and practices as described in the introduction. It is believed that many Greek philosophers (Plato, for instance) were deeply influenced by Pythagoras, so much so that Bertrand Russell felt that he should be considered one of the most influential Western philosophers.

We will return to the Pythagorean School after we take a detour and look at contributions outside the Greek world (this is often difficult for many Eurocentric historians to swallow!)

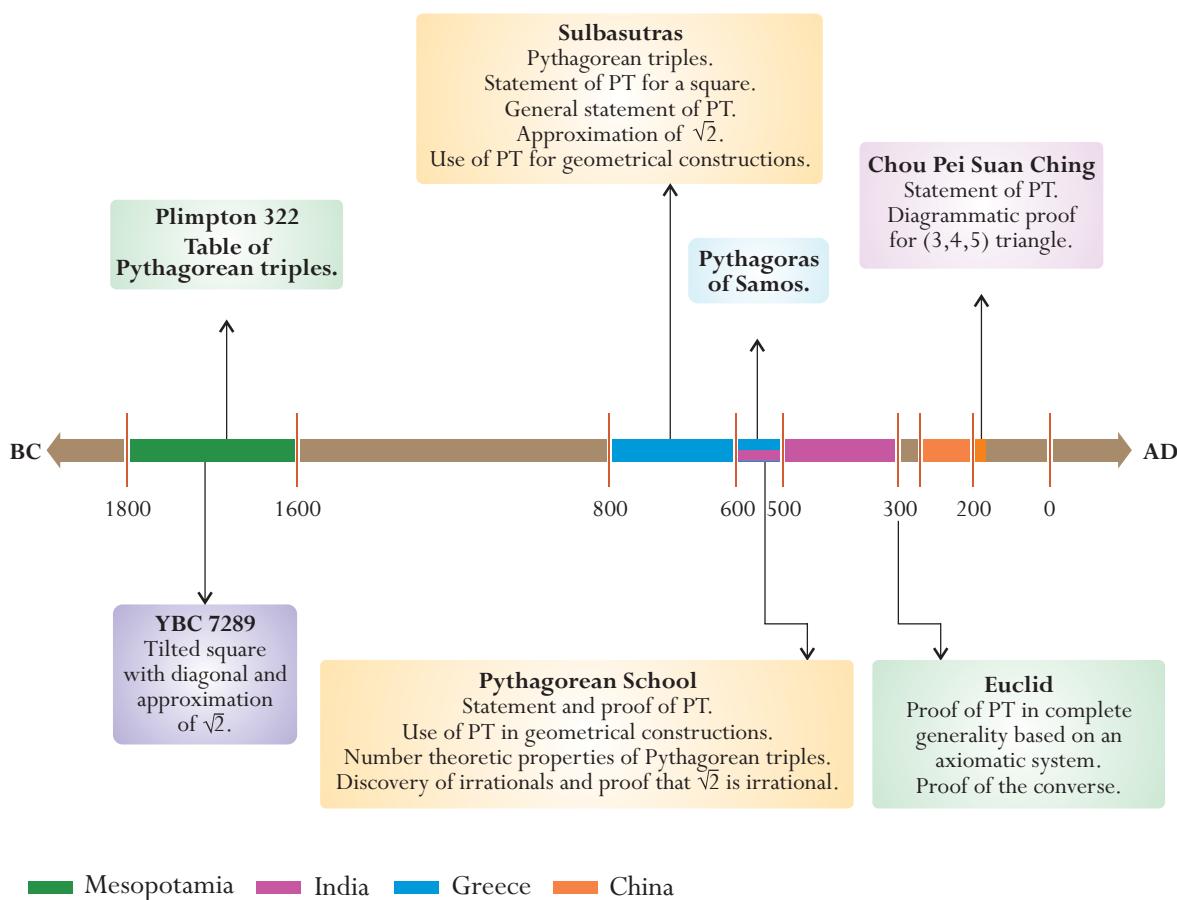
## The problem of dating!

As students of history we must realise that the greatest challenge historians of antiquity face is that of giving accurate dates to events. There are many reasons for this, including the fact that many cultures were oral, records of events were burnt, languages of some cultures have yet to be deciphered and very often, as mentioned earlier, our only knowledge about people and events are from historians referring to them many years later. So I have tried to use a very conservative and broad timeline.

### Pythagoras Theorem (PT)

*In right-angled triangles, the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle.*

## Timeline of the History of the Pythagoras Theorem (PT)



Let us now try and understand the history of PT chronologically.

### The Mesopotamian contribution

You may recall from your school history that one of the oldest known civilizations (Mesopotamia or Babylonia) existed in the geographical region between the Tigris and Euphrates rivers. Records of this civilization date back to 3500 BC. They used the sexagesimal system (base 60) and used mathematics for record keeping and astronomy. They also seemed interested in number theory and geometry.

We know this because they left records of their work on thousands of clay tablets, five hundred of which seem mathematical in nature. There are two main sources that tell us about the Mesopotamian contribution to the PT. These are clay tablets

with wedge shaped markings on them. Historians date these to the period of Hammurabi between 1800 BC and 1600 BC.

The tablet known as YBC 7289 (tablet number 7289 from the Yale Babylonian Collection) shows a tilted square with wedged shaped markings. The markings show calculations for the approximation of  $30\sqrt{2}$ . This would not have been possible

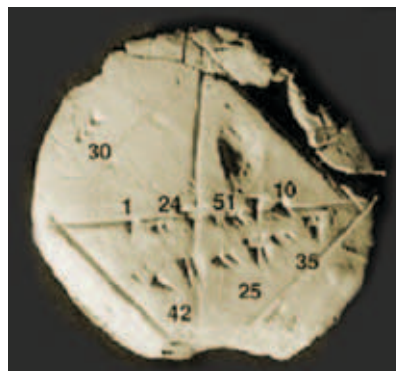


Fig. 1, YBC 7289



Fig.2, Plimpton 322

without the knowledge of the PT or at least the special case of the isosceles right-triangle (see Figure 5).

The second is referred to as Plimpton 322, a slightly damaged clay tablet measuring 13 9 2 cm and a part of the University of Columbia collection. It contains a 15 4 table of numbers. The table is thought to be a list of Pythagorean triples. Pythagorean triples are integers  $(a, b, c)$  which satisfy the equation  $a^2 + b^2 = c^2$  (see articles on Pythagorean triples). For example  $(3, 4, 5)$  is a Pythagorean triple.

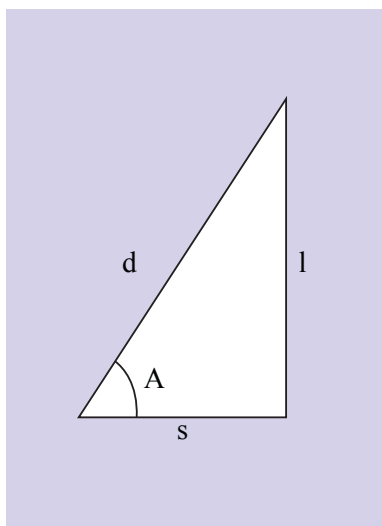


Fig. 3

Let us look at the entries in the tablet. The tablet contains errors in Rows 2, 9, 13 and 15 and the original entries were in base 60, but the table below is in base 10 with the errors corrected. Here 's' stands for the shortest side of a right-triangle, 'd' for the hypotenuse and 'l' for the other side (see Figure 3).

Let us look at entries in the first row. It is not hard to check that  $169^2 - 119^2 = 120^2$ . That is,  $(119, 120, 169)$  is a Pythagorean triple. Similarly we can verify that  $(s, l, d)$  form Pythagorean triples in each row (if you are sceptical – go ahead and do the computations!) So it seems clear

that the tablet was a list of Pythagorean triples.

However, mathematical historians are left with many questions. What exactly does Column 1 represent? Is there any pattern behind the choice of 's' and 'd'? Is there some general principle at work here?

There are three main interpretations of the purpose of these tablets. The first is that Plimpton 322 is a trigonometric table of some sort. Column 1 is  $\text{Csc}^2 A$ , where angle 'A' ranges from just above  $45^\circ$  to  $58^\circ$ .

$(d/l)^2$	s	d	
$(169/120)^2$	119	169	Row 1
$(4825/3456)^2$	3367	4825	Row 2
$(6649/4800)^2$	4601	6649	Row 3
$(18541/13500)^2$	12709	18541	Row 4
$(97/72)^2$	65	97	Row 5
$(481/360)^2$	319	481	Row 6
$(3541/2700)^2$	2291	3541	Row 7
$(1249/960)^2$	799	1249	Row 8
$(769/600)^2$	481	769	Row 9
$(8161/6480)^2$	4961	8161	Row 10
$(75/60)^2$	45	75	Row 11
$(2929/2400)^2$	1679	2929	Row 12
$(289/240)^2$	161	289	Row 13
$(3229/2700)^2$	1771	3229	Row 14
$(106/90)^2$	56	106	Row 15

Table 1 - Plimpton 322 in modern notation



Row#	p	q	x	y	S	d
1	12	5	144/60	25/60	119	169
8	32	15	128/60	101250/ 216000	799	1249
15	9	5	108/60	20/36	56	106

Table 2

The second and third interpretations are more involved and we will require another table to understand them! (See Table 2; the roles played by 'x' and 'y' will become clear after reading the following two paragraphs.)

Let us examine the first, last and middle rows of Plimpton 322 to see if any pattern emerges. Neugebauer and others have proposed that 's' and 'd' are generated by a pair of positive integers 'p' and 'q', which are of opposite parity and relatively prime. The relationship between s, d, l, p and q is as follows:  $s = p^2 - q^2$ ,  $d = p^2 + q^2$ ,  $l = 2pq$ . In fact, if we are given any two integers p, q relatively prime, with one of them even, we can generate all Pythagorean triples. Quite a remarkable feat, don't you think? The article by S Shirali (elsewhere in this issue) explores various ways of generating Pythagorean triples and in the next issue this method will be explored in detail.

Of course one can now ask, is there a pattern in the choice of p and q (and also x and y)? They are so-called 'regular' numbers (numbers of the form  $2^a 3^b 5^c$ , where a, b, c are integers). Can you see the connection between 2, 3, 5 and the sexagesimal system? Moreover, there is a pattern on how p, q, x, y change as we move from Row 1 to 15; but this is quite technical, so for more details we refer the reader to [AA] and [RE].

The third interpretation, first put forward by Bruins in 1949, is called the 'reciprocal' method. Here the table is believed to be generated by a pair of rational numbers 'x' and 'y' such that  $xy = 1$  (So x and y are a pair of reciprocals.)

$$\text{Here } \frac{s}{l} = \frac{x - y}{2} \text{ and } \frac{d}{l} = \frac{x + y}{2}$$

It is impossible to say for sure which interpretation is correct. But scholars feel that apart from Plimpton 322 there is no other evidence of knowledge of trigonometry in Mesopotamia, and that the second interpretation is not in keeping with the approach to mathematics found in the other

tablets. Many scholars favor the 'reciprocal' method as they feel that it is not only mathematically valid, but is also historically, archeologically and linguistically consistent with the style and conventions of ancient Babylonian mathematics.

It may amuse readers to know that scholars like Robson [RE] feel that the author of Plimpton 322 was a teacher, and the tablet is a kind of 'question bank' which would "have enabled a teacher to set his students repeated exercises on the same mathematical problem, and to check their intermediate and final answers without repeating the calculations himself."

These two tablets, along with evidence from tablets found in Susa and Israel from the Babylonian period, clearly demonstrate that they were well versed with the PT and were also adept at using it.

### Contribution from India

The history of India and Indian mathematics poses many challenges to historians. The difficulties range from giving a balanced and accurate picture to dating various events. At the same time, there seems to be a great deal of interest today in the contributions of the Indian subcontinent to mathematics. This is particularly so after the discovery of the Kerala School of mathematics, which came very close to discovering calculus long before Newton and Leibnitz. Mathematics in India was inspired by astronomy, record keeping, religion and perhaps sheer curiosity.

Historians believe that early Indian civilizations date to the third or fourth millennium BC. Our main interest is in the *Sulbasutras*, which literally means the 'rule of cords.' They are a series of texts (*Vedangas*) which accompany the Vedas and give detailed instructions on how rituals are to be performed and sacrificial altars (*Vedis*) constructed.

The most important Sulbasutras are attributed to Boudhayana, Manava, Apastamba and Katyayana. Boudhayana is believed to have lived around 800 BC and Apasthamba around 500 BC.

What is of significance is that in the Sulbasutras we find a general statement of the PT as follows [see PK]: "The cord [equal to] the diagonal of an

*oblong makes [the area] that both the length and width separately [make]. By knowing these [things], the stated construction is [made].* " {Apastamba Sulbasutra 1.4 and Boudhayana Sulbasutra 1.12}

This may be the earliest general statement (at least for right triangles with rational sides) of the PT, perhaps predating 800 BC. However, we must mention that there is a reference to this result in China clearly before Pythagoras, but whose exact date is unknown. We will discuss this in the section on the Chinese contribution.

We also find in the Sulbasutras (see [PK]) the application of the PT to a square (isosceles right triangle): *"The cord [equal to] the diagonal of a [square] quadrilateral makes twice the area. It is the doubler of the square."* {Apastamba Sulbasutra 1.6, Boudhayana Sulbasutra 1.9 and Katyayana Sulbasutra 2.9}

We will discuss in a later section how perhaps the recognition of PT for the special case of the isosceles right triangle led to the discovery of the general theorem.

The Sulbasutras also contain Pythagorean triples, approximation of square roots and the use of PT for many geometrical constructions. Why don't you try your hand (using a straightedge and compass) at some of the constructions found in the Sulbasutras? For example, try constructing a square whose area is equal to the sum of two given squares. Or, try constructing a square whose area is equal to that of a given rectangle.

What emerges clearly is that the Sulbakaras (authors of the sutras) had a very good understanding of PT and its applications, both to extracting roots and to geometrical constructions. We must, however, acknowledge that there is no evidence that the notion of *proving* mathematical statements was part of their framework.

### Chinese contribution

We are all aware that China has been home to a very ancient civilization that developed along the rivers of Yangtze and Huang Ho more than 5000 years ago. The Chinese were interested in many areas of mathematics, again perhaps driven by astronomy, the need to have accurate calendars and sheer intellectual interest.

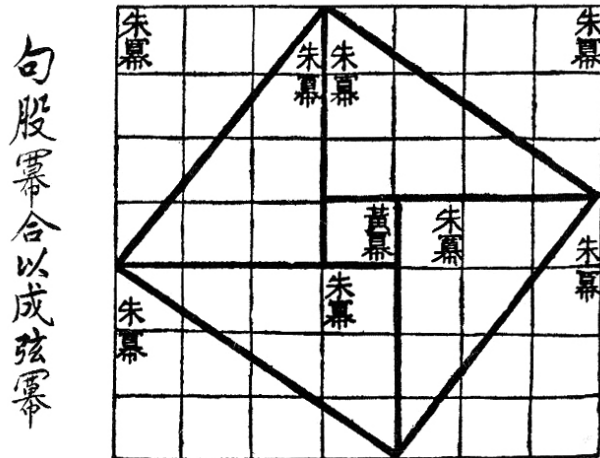


Fig. 4 - Chou Pei Suan Ching

As far as the PT is concerned, our main source of information is the Chou Pei Suan Ching (The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven). The exact date of this book has been debated for a long time. It refers to a conversation between the Duke Zhou Gong and his minister Shang Kao around 1000 BC, discussing the properties of a right triangle, with a statement of the PT and a diagrammatic proof given. It is not clear if such a conversation did take place. However, scholars believe that earlier results were put together in the form of a book, from 235 BC to 156 BC, and were edited by Zhang Chang around 156 BC. Further, a famous mathematician Zhao Shuang wrote commentaries on the Chou Pei, adding original material of his own, including the well-known diagrammatic proof (see Figures 4, 7 and 8).

The PT in Chinese literature is referred to as 'kou ku' (see [JG]). We will discuss the diagrammatic proof in the section on how the Greeks might have arrived at a general proof for the PT. What is clear is that the Chinese were not only aware of PT long before Pythagoras, but had many applications for it and came up with a pictorial demonstration for the (3, 4, 5) case which can be generalized.

### Returning to the Greeks

Having traversed the globe, let us return to the Greek contribution. There is no doubt that the Greeks were the first to bring in the notion of proof in mathematics. The Pythagorean School seems to have definitely had a proof for PT at least

for rational sides. Stephen Hawking argues in [HS] that they had perhaps an incomplete proof for the general theorem, because Euclid took great pains to give a new and complete proof in the Elements. The Pythagorean School were also the first to prove rigorously that the square root of 2 is irrational. While earlier civilizations did come up with approximations for  $\sqrt{2}$  there is no clear evidence that they were aware of irrationals.

Before we conclude the Greek contribution we should mention Euclid's role (approx 300 BC). As you perhaps know, he is the author of *Elements*, a collection of 13 books containing 465 propositions from plane geometry, number theory and solid geometry. He was the first person to create an axiomatic framework for mathematics with rigorous proofs. Once again we know very little of Euclid, except that he worked in the great library of Alexandria during the rule of Ptolemy I (323 – 283 BC). Euclid gave two rigorous proofs of PT: one is the 47<sup>th</sup> proposition of Book I and the other is the 31<sup>st</sup> proposition of Book VI.

Proposition VI.31 is a generalization of PT, for while Proposition I. 47 refers to squares constructed on the three sides of a right-angled triangle, Proposition VI.31 refers to any figure constructed similarly on the sides of a right-triangle. For example, if semicircles are constructed on

the sides of a right-triangle, then the area of the semicircle on the diagonal is equal to the sum of the areas of the semicircles on the other two sides. He was also the first to give a rigorous proof of the converse of PT (proposition 48 of Book I).

Here is a lovely compliment (sourced from [HS]) a fellow Greek, Proclus, pays to Euclid several centuries later: *"If we listen to those who wish to recount ancient history, we may find some of them referring this theorem (PT) to Pythagoras and saying that he sacrificed an ox in honour of his discovery. But for my part, while I admire those who first observed the truth of this theorem, I marvel more at the writer of the Elements, not only because he made it fast by a most lucid demonstration, but because he compelled assent to the still more general theorem by the irrefragable arguments of science in the sixth Book. For in that Book he proves generally that, in right-angled triangles, the figure on the side subtending the right angle is equal to the similar and similarly situated figures described on the sides about the right angle."*

### What motivated the discovery of PT?

It is a matter of great curiosity as to how human beings all over the world discovered a result such as the PT. There are two main threads of speculation.

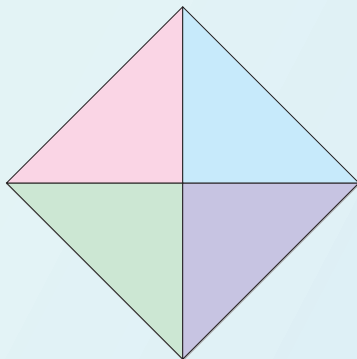


Fig. 5

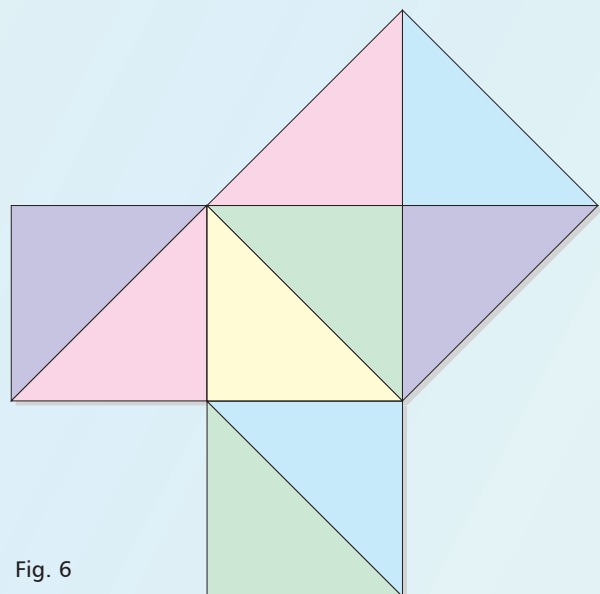


Fig. 6

The first thread looks at the special isosceles right triangle. Historians believed that tiles as shown in Figure 5 would have been the inspiration for the mathematically curious.

If that figure is not self evident to you, what about Figure 6?

The second thread looks at the triple (3, 4, 5). There is evidence that the Egyptians knew the relation  $3^2 + 4^2 = 5^2$ . However, there is no evidence that they knew that a triangle with side lengths 3, 4 and 5 units was right-angled. Stephen Hawking and others speculate that the next jump in ideas took place when there was a realisation, much along the lines of the Chinese, that a right triangle with legs of length 3 and 4 has hypotenuse of length 5, essentially proving PT for this special case.

Let us see how this is done. Start with a right-angled triangle of side lengths 3 and 4. You then wrap around 4 such right-triangles to form a  $7 \times 7$  square (see Figures 7 and 8).

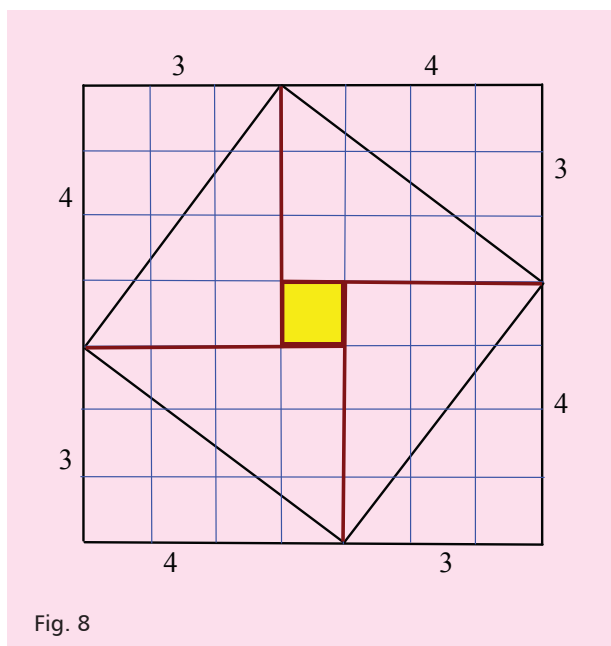
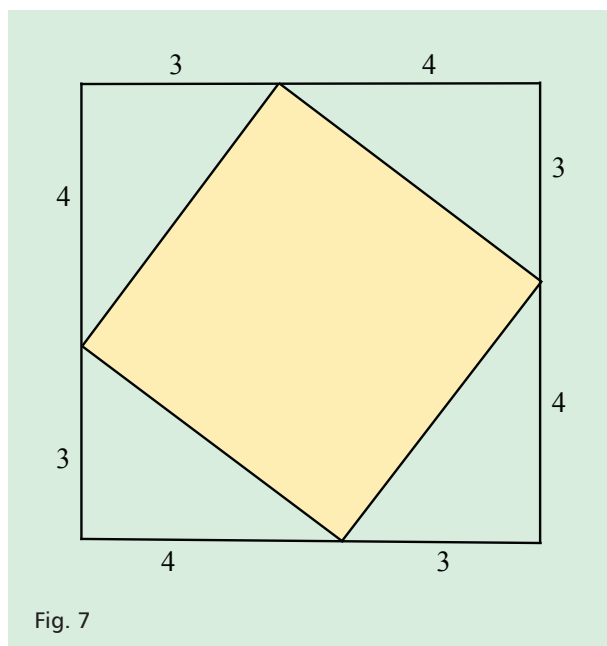
Now look at the inner square that is sitting on the hypotenuse of each of the four triangles.

There are two ways to see that it is 25 square units. One is that the original square is 49 square units and it is made up of two  $3 \times 4$  rectangles

and the inner square. The other is that the inner square is made of two  $3 \times 4$  rectangles and a unit square. Hence we have shown that a triangle with legs of size 3 and 4 units has a hypotenuse of size 5 and the PT holds for this triangle.

This method can be generalized to other right-angled triangles with sides of integer lengths. For example, take a right-angled triangle whose legs are of length 5 and 12. Then using the method above, one will get a  $17 \times 17$  square with an inner square of size  $13 \times 13$ . This shows that the hypotenuse of such a triangle is 13 units. And using the fact that  $5^2 + 12^2 = 13^2$ , we have once again a specific example of the PT. The above figure can also be used to establish PT for any right-angled triangle. Can you use algebra and prove it for yourself? This is essentially how Bhaskara proved PT in the eleventh century AD.

It is not clear if Pythagoras and others used the method I have just asked you to prove. As mentioned earlier, since Euclid gave a completely different proof, historians believe that Pythagoras might have used the method of similar triangles to establish PT. However since they dealt only with rational numbers, this proof would have been incomplete.





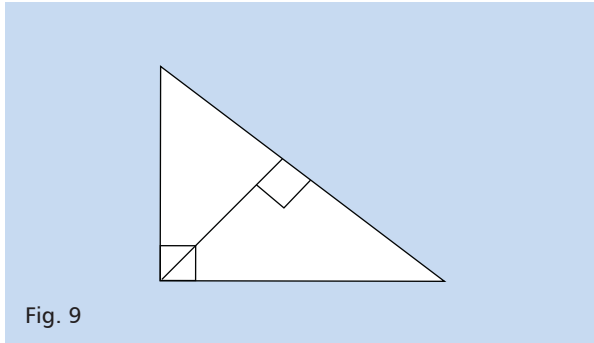


Fig. 9

Are you familiar with the proof of PT using similar triangles? If not, why don't you give it a try? Figure 9 will help you. It is considered the shortest proof of the Pythagoras theorem!

### Endnote

I hope in the course of reading this article you have got a sense of the rich history and depth behind the Pythagoras Theorem and how challenging ancient history is. You probably have also realized that it is a quirk of fate that has named the most famous theorem after Pythagoras. It well might have been the 'Mitharti siliptim (Square of the diagonal) Theorem' from Mesopotamia or the 'Sulba Theorem' from India or the 'Kou ku Theorem' from China! So, what is in a name?

### References

1. [AA] Abdulaziz, Abdulrahman A, *The Plimpton322 Tablet and the Babylonian Method of Generating Pythagorean Triples*, 2010 (source: arXiv:10004.0035v1[math.HO]).
2. [DW] Dunham, William, *Journey Through Genius: The Great Theorems of Mathematics*, John Wiley & Sons, New York, 1990.
3. [EH] Eves, Howard, *An Introduction to the History of Mathematics* (fourth edition), Holt, Rinehart and Winston, New York, 1976.
4. [HS] Hawking, Stephen, *God Created the Integers: the Mathematical Breakthroughs that Changed History*, Running Press, Philadelphia 2005.
5. [JG] Joseph, George Gheverghese, *The Crest of the Peacock: Non-European Roots of Mathematics*, Penguin Books, New York 1991.
6. [ME] Maor, Eli, *The Pythagorean Theorem: a 4,000-year History*, Princeton University Press, Princeton 2007.
7. [PK] Plofker, Kim, *Mathematics in India: From 500 BCE to 1800 CE*, Princeton University Press, Princeton 2009.
8. [RE] Robson, Eleanor, *Words and Pictures: New Light on Plimpton 322*, *American Mathematical Monthly*, 109, 105-120, 2002.
9. [YS] Yao, Shing-Tung, *The Past, Present and Future of Mathematics in China and India*, (source: homepage.ntu.edu.tw/~d95221004/math\_history.pdf)

### Sources for Figures

- Figure 1(YBC 789): [http://en.wikipedia.org/wiki/YBC\\_7289#History](http://en.wikipedia.org/wiki/YBC_7289#History)
- Figure 2 (Plimpton 322): [http://en.wikipedia.org/wiki/Plimpton\\_322](http://en.wikipedia.org/wiki/Plimpton_322)
- Figure 4(Chou Pei Suan Ching): [http://en.wikipedia.org/wiki/Chou\\_Pei\\_Suan\\_Ching](http://en.wikipedia.org/wiki/Chou_Pei_Suan_Ching)



SHASHIDHAR JAGADEESHAN received his PhD from Syracuse University in 1994. He has been teaching mathematics for the last 25 years. He is a firm believer that mathematics is a very human endeavour and his interest lies in conveying the beauty of mathematics to students and also demonstrating that it is possible to create learning environments where children enjoy learning mathematics. He is the author of Math Alive!, a resource book for teachers, and has written articles in education journals sharing his interests and insights. He may be contacted at [jshashidhar@gmail.com](mailto:jshashidhar@gmail.com)