How to Prove it

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In this episode of "How To Prove It", we prove a beautiful and striking formula first found by Leonhard Euler; it gives the area of the pedal triangle of a point with reference to another triangle.

Euler's formula for the area of a pedal triangle

Given a triangle ABC and a point P in the plane of ABC (note that P does not have to lie within the triangle), the **pedal triangle** of P with respect to $\triangle ABC$ is the triangle whose vertices are the feet of the perpendiculars drawn from P to the sides of ABC. See Figure 1. The pedal triangle relates in a natural way to the parent triangle, and we may wonder whether there is a convenient formula giving the area of the pedal triangle in terms of the parameters of the parent triangle. The great 18th-century mathematician Euler found just such a formula (given in Box 1). It is a compact and pleasing result, and it expresses the area of the pedal triangle in terms of the radius R of the circumcircle of $\triangle ABC$ and the distance between P and the centre O of the circumcircle.

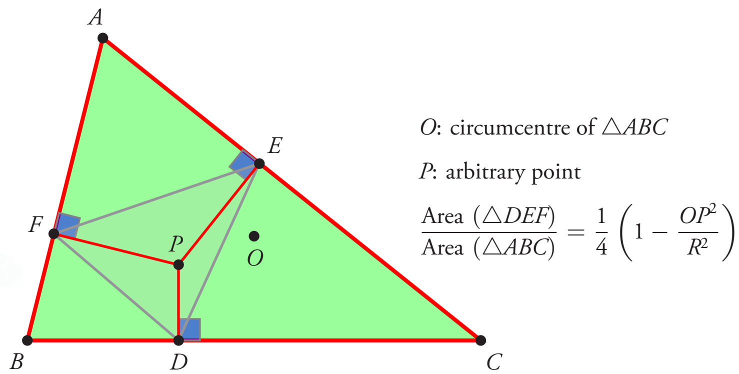


Figure 1. Euler's formula for the area of the pedal triangle of an arbitrary point

Keywords: Circle theorem, pedal triangle, power of a point, Euler, sine rule, extended sine rule, Wallace-Simson theorem

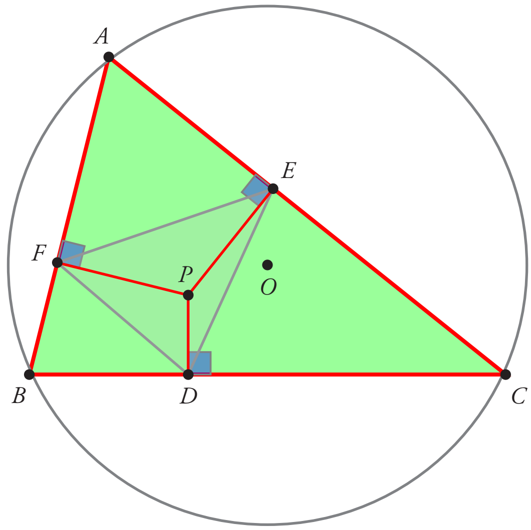


Figure 2. Proof of Euler's formula – step I

Euler's formula for the area of a pedal triangle

Theorem 1 (Euler)**.** Given a reference triangle ABC and a point P in the plane of the triangle, the ratio of the area of the pedal triangle of P to the area of $\triangle ABC$ is given by:

$$\frac{\text{Area of } \triangle DEF}{\text{Area of } \triangle ABC} = \frac{1}{4}\left(1 - \frac{OP^2}{R^2}\right), \tag{1}$$

where O is the circumcentre and R is the radius of the circumcircle of $\triangle ABC$.

Box 1

The occurrence of the distance OP in this formula comes as a major surprise. The reader is invited to look for a proof before reading on. It is a pretty challenge!

Proof of Euler's formula. The proof will unfold in several stages. The sine formula for area (see Figure 2) tells us that

$$\text{Area of } \triangle DEF = \frac{1}{2}\left(DE \cdot DF \cdot \sin \measuredangle EDF\right). \tag{2}$$

We now obtain simplified expressions for each term on the RHS of this formula: DE, DF and $\sin \measuredangle EDF$. This will yield the desired result.

First, consider DE. Note that $DCEP$ is a cyclic quadrilateral, CP being a diameter of its circumcircle (to see why, note that $\measuredangle PDC = 90^\circ = \measuredangle PEC$). The 'extended sine rule' (i.e., the statement that in any triangle, the ratio of each side to the sine of the opposite angle equals the diameter of the circumcircle of the triangle) applied to $\triangle CDE$ tells us that

$$\frac{DE}{\sin \measuredangle ECD} = CP, \tag{3}$$

and so:

$$DE = CP \cdot \sin C. \tag{4}$$

In exactly the same way, we get:

$$DF = BP \cdot \sin B. \tag{5}$$

Next, we seek an expression for $\sin \measuredangle EDF$. For this, we extend BP till it meets the circumcircle again at point K; draw segments CK, AK (see Figure 3).

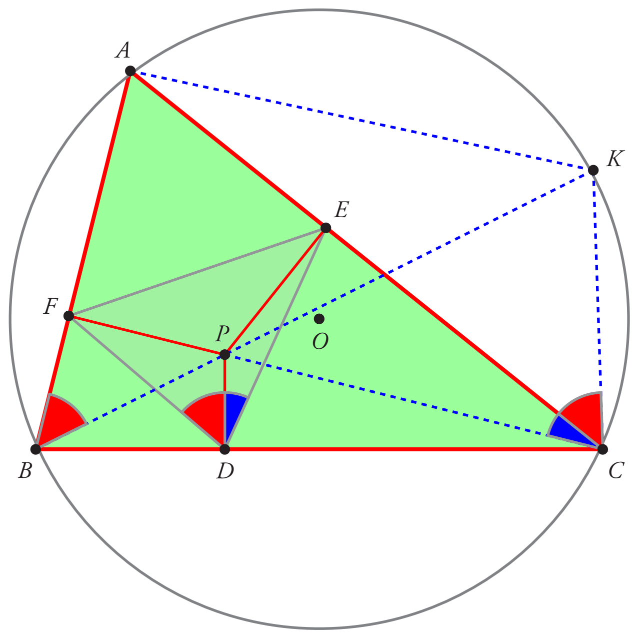


Figure 3. Proof of Euler's formula – step II

First we note that $\measuredangle EDF = \measuredangle KCP$ (as marked in the figure); for: $\measuredangle PDE = \measuredangle PCE$ and $\measuredangle PDF = \measuredangle PBF = \measuredangle KBA = \measuredangle KCA$. Hence we get by addition: $\measuredangle EDF = \measuredangle KCP$, and so: $\sin \measuredangle EDF = \sin \measuredangle KCP$. Next:

$$\frac{PK}{\sin \measuredangle KCP} = \frac{CP}{\sin \measuredangle PKC} = \frac{CP}{\sin A}. \tag{6}$$

This yields:

$$\sin \measuredangle KCP = \frac{PK \cdot \sin A}{CP} = \sin \measuredangle EDF. \tag{7}$$

Hence:

$$\begin{aligned} \text{Area of } \triangle DEF &= \frac{1}{2} CP \cdot \sin C \cdot BP \cdot \sin B \cdot \frac{PK \cdot \sin A}{CP} \\ &= \frac{1}{2} BP \cdot PK \cdot \sin A \, \sin B \, \sin C. \end{aligned} \tag{8}$$

Now we invoke another well-known formula for the area of a triangle which follows from the extended sine rule and the sine formula for area:

$$\begin{aligned}\text{Area of } \triangle ABC &= \frac{1}{2} bc \sin A \\ &= \frac{1}{2} 2R \sin B \cdot 2R \sin C \cdot \sin A \\ &= 2R^2 \sin A \sin B \sin C.\end{aligned}$$

Hence we obtain:

$$\text{Area of } \triangle DEF = \frac{1}{2} BP \cdot PK \cdot \frac{\text{Area of } \triangle ABC}{2R^2}. \qquad (9)$$

Next, note that $BP \cdot PK$ is simply the **power** of the point P with respect to the circumcircle, and this is equal to $R^2 - OP^2$. (Some of you may not recall the definition of "power of a point with respect to a given circle". For your convenience, we have assembled all the relevant concepts and formulas in the appendix at the end of this article.) Hence:

$$\frac{\text{Area of } \triangle DEF}{\text{Area of } \triangle ABC} = \frac{R^2 - OP^2}{4R^2} = \frac{1}{4}\left(1 - \frac{OP^2}{R^2}\right), \qquad (10)$$

as claimed.

A corollary to Euler's formula: the Wallace-Simson theorem. If P lies on the circumcircle, then $OP = R$, so the formula tells us that the area of the pedal triangle is 0. This is equivalent to asserting that the vertices of the pedal triangle lie in a straight line. In this form, the statement is well-known as the Wallace-Simson theorem (see Figure 4):

Theorem 2 (Wallace & Simson)**.** The feet of the perpendiculars dropped from a point on the circumcircle of a triangle to the sides of the triangle lie in a straight line.

The Wallace-Simson theorem can be proved directly, by old-fashioned "angle-chasing". (Try to find a proof on your own! Or see Appendix 2.) The line on which points D, E, F lie is called the **pedal line** of P (or, in older texts, the **Simson line** of P).

Appendix 1: Power of a Point

Let Γ be a circle with centre O, and let P be any point in the plane of the circle. To start with, consider the case when P lies outside the circle. Let ℓ be any line through P, and let ℓ cut the circle at points A, B. (See Figure 5.) Let PT be a tangent from P to the circle. Then $PA \cdot PB = PT^2$. To see why, we only need to see that $\triangle PAT \sim \triangle PTB$ (compare the angles of the triangles to see why). By the theorem of Pythagoras, $PT^2 = OP^2 - r^2$. Hence $PA \cdot PB = OP^2 - r^2$. Since the value of $OP^2 - r^2$ does not depend on the choice of line ℓ, it follows from this that the product $PA \cdot PB$ is independent of ℓ; it is constant over all such lines.

If P lies within the circle, a different figure needs to be drawn; the tangent PT now does not exist, but the conclusion remains the same: the product $PA \cdot PB$ is independent of the choice of line ℓ, and the constant value is equal to $OP^2 - r^2$. The reader is invited to draw the relevant figure in this case and to verify the stated conclusion.

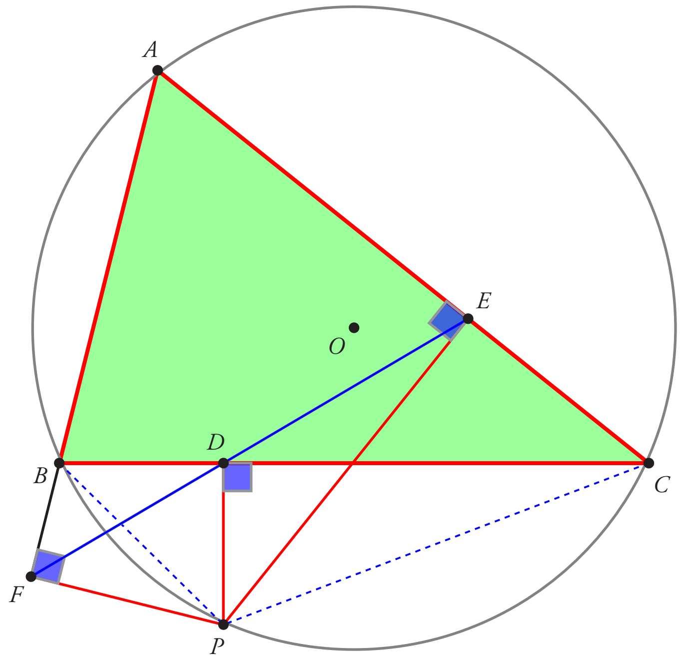


Figure 4. The Wallace-Simson theorem: the feet of the perpendiculars from P to the sidelines of the triangle lie in a straight line

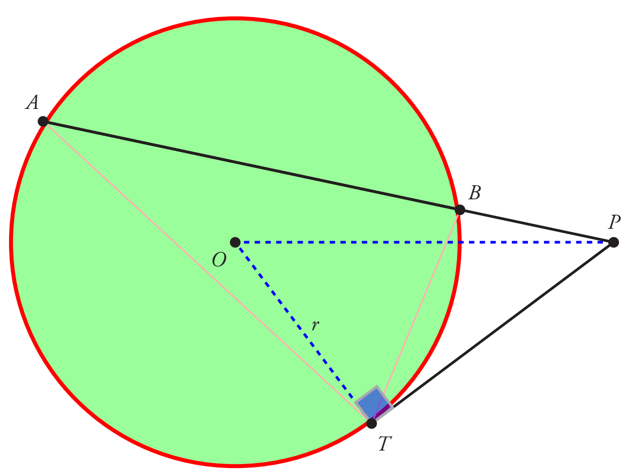


Figure 5.

If P lies outside the circle (as in Figure 5), then $OP > r$, hence $PA \cdot PB > 0$. If P lies inside the circle, then $OP < r$, hence $PA \cdot PB < 0$. And if P lies on the circle, then $OP = r$, hence $PA \cdot PB = 0$.

The quantity $OP^2 - r^2$ is called the **power of point P with respect to the circle**. Thus the power is positive for points P lying outside the circle, 0 for points P on the circle, and negative for points P lying inside the circle.

Appendix 2: Proof of the Wallace-Simson theorem

In order to prove that points F, D, E lie in a straight line, we must prove that $\measuredangle PDF + \measuredangle PDE = 180°$. (See Figure 6.) Now $\measuredangle PDF = \measuredangle PBF$ (from the cyclic quadrilateral $PDBF$); and $\measuredangle PDE + \measuredangle PCE = 180°$ (from the cyclic quadrilateral $PCED$; to see why it is cyclic, note that $\measuredangle CDP$ and $\measuredangle CEP$ are both right angles). So the task comes down to proving that $\measuredangle PBF = \measuredangle PCE$. But this readily follows from the fact that quadrilateral $ABPC$ is cyclic: $\measuredangle PBF$ is an exterior angle of this quadrilateral, and it is equal to the interior opposite angle which is $\measuredangle PCA$.

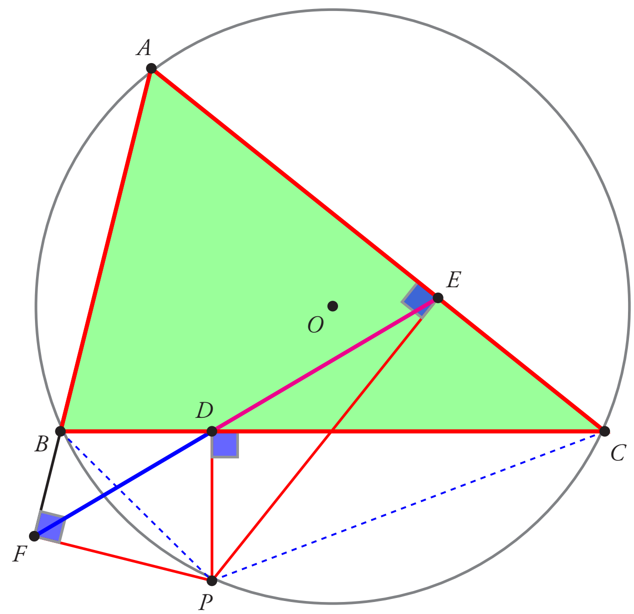


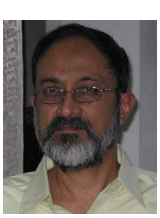
Figure 6.

References

1. A. Bogomolny, "Sides and Area of Pedal Triangle" from *Interactive Mathematics Miscellany and Puzzles*, http://www.cut-the-knot.org/triangle/PedalTriangle.shtml, Accessed 03 January 2016

2. R. A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, 1960, 135–141

3. Weisstein, Eric W. "Pedal Triangle." From *MathWorld*–A Wolfram Web Resource. http://mathworld.wolfram.com/PedalTriangle.html



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