Using Fractions, Odd Squares, Difference of Two Squares

How to Generate Pythagorean Triples -1

feature

Exploring different generative methods

Generating Primitive Pythagorean Triples can introduce students to number theoretic properties, enhance logical reasoning and encourage students to find answers to their 'whys'.

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The relation $a^2+b^2=c^2$ is so familiar to us that we often quote it without saying what a, b, c represent! And this, no doubt, is because the Pythagorean theorem is so well known. We know that if a, b, c are the sides of a right angled triangle, with c as the hypotenuse, then $a^2+b^2=c^2$. We also know, conversely, that if a, b, c are positive numbers which satisfy this relation, then one can construct a right angled triangle with legs a, b and hypotenuse c. Because of this association, we call a triple (a, b, c) of positive integers satisfying this relation a ***Pythagorean triple***, PT for short. But such triples have additional properties of interest that have nothing to do with their geometric origins; they have *number theoretic properties*, and we will be studying some of them in this and some follow up articles.

The most well known PT is the triple (3, 4, 5). Since its numbers are coprime — i.e., there is no factor exceeding 1 which divides all three of them — we call it 'primitive', and the triple is called a ***primitive Pythagorean triple*** or PPT for short. In this article we explore some ways of generating PPTs.

Note. Throughout this article, when we say 'number' we mean 'positive integer'. If we have some other meaning in mind, we will state it explicitly.

What is a 'number theoretic' property?

Before proceeding we must state what we mean by a 'number theoretic' property. Below, we list six such properties about numbers. On examining them you should be able to make out what is meant by the phrase 'number theoretic property'. (We have not justified the statements; we urge you to provide the proofs.)

1. *The sum of two consecutive numbers is odd.*
2. *The sum of two consecutive odd numbers is a multiple of 4.*
3. *The sum of three consecutive numbers is a multiple of 3.*
4. *An odd square leaves remainder 1 when divided by 8.*
5. *The square of any number is either divisible by 3, or leaves remainder 1 when divided by 3.*
6. *The sum of the first n odd numbers is equal to n^2.*

In contrast, here are some statements which are true for any kind of quantity, not just for positive integers: *For any two quantities a and b we have*:

$$a^2 - b^2 = (a-b)\cdot(a+b),$$

$$a^3 - b^3 = (a-b)\cdot(a^2+ab+b^2),$$

$$a^2 + b^2 \geq 2ab.$$

These statements are true even if a and b are not integers. But statements (1)–(6) presented earlier have no meaning if the numbers involved are not integers.

Generating PPTs

We have already given(3, 4, 5) as an example of a PPT. How do we generate more such triples? Below we describe four ways of doing so. The first three are presented without justification; we do not show how we got them, but they are fun to know! In the case of the fourth one, we derive it in a logical way.

Method #1: Using Odd Squares

This method is often found by students who like to play with numbers on their own, and it is perhaps the simplest way of generating Pythagorean triples.

Select any odd number $n > 1$, and write n^2 as a sum of two numbers a and b which differ by 1 *(here b is the larger of the two numbers); then (n, a, b) is a PT; indeed, it is a PPT.*

Examples

- Take $n=3$; then $n^2=9=4+5$, so $a=4$, $b=5$. The triple is (3, 4, 5).
- Take $n=5$; then $n^2=25=12+13$, so $a=12$, $b=13$. The triple is (5, 12, 13).
- Take $n=7$; then $n^2=49=24+25$, so $a=24$, $b=25$. The triple is (7, 24, 25).
- Observe that each triple generated here has the form $(n, a, a+1)$ where $2a+1=n^2$.

Exercises.

(1.1) Justify why this procedure yields PTs.
(1.2) Justify why these PTs are PPTs.
(1.3) Find a PPT which cannot be generated by this method.

Method #2: Using Unit Fractions With Odd Denominator

Of all the methods one generally sees, this one is perhaps the strangest!

Let n be any odd number. Compute the sum $\frac{1}{n}+\frac{1}{n+2}$ and write it in the form $\frac{a}{b}$ where a, b are coprime. Then (a, b, b+2) is a PPT.

Here are some PPTs generated using this method.

- Take $n=1$; then $n+2=3$, and $\frac{1}{1}+\frac{1}{3}=\frac{4}{3}$. We get the PPT (4, 3, 5).
- Take $n=3$; then $n+2=5$, and $\frac{1}{3}+\frac{1}{5}=\frac{8}{15}$. We get the PPT (8, 15, 17).
- Take $n=5$; then $n+2=7$, and $\frac{1}{5}+\frac{1}{7}=\frac{12}{35}$. We get the PPT (12, 35, 37).
- Take $n=7$; then $n+2=9$, and $\frac{1}{7}+\frac{1}{9}=\frac{16}{63}$. We get the PPT (16, 63, 65).

Exercises.

(2.1) Justify why this yields PTs.
(2.2) Explain why these PTs are PPTs.
(2.3) Find a similar method that uses the even positive integers.
(2.4) Find a PPT which cannot be generated by this method.

Method #3: Using Mixed Fractions

In the same way that we used unit fractions we may also use mixed fractions. We write the following sequence of mixed fractions:

$$1\tfrac{1}{3},\quad 2\tfrac{2}{5},\quad 3\tfrac{3}{7},\quad 4\tfrac{4}{9},\quad 5\tfrac{5}{11},\quad 6\tfrac{6}{13},\quad ....$$

The pattern behind the sequence should be clear. Now we write each fraction in the form $\frac{a}{b}$; i.e., we write each one as an 'improper' fraction. We get:

$$\frac{4}{3},\quad \frac{12}{5},\quad \frac{24}{7},\quad \frac{40}{8},\quad \frac{60}{11},\quad \frac{84}{13},\quad ....$$

Examining these fractions, we see quickly that if $\frac{a}{b}$ is a fraction in the sequence, then $(b,\ a,\ a+1)$ is a PPT. So we get the following PPTs:

$(3,4,5)$, $(5,12,13)$, $(7,24,25)$, $(9,40,41)$, $(11,60,61)$, $(13,84,85)$,

.... Strangely, we have obtained the same PPTs that we got with the first method.

Exercises.

(3.1) Justify why this yields PTs.
(3.2) Explain why it yields the same PTs that we obtained by Method #1.
(3.3) Explain why these PTs are PPTs.
(3.4) Find a PPT which cannot be generated by this method.

Remark

All these are 'ad hoc' methods; in no case do we give any hint as to how we got the method. In contrast, here is a method which we actually derive. And that is surely so much more satisfactory.

Method #4: Using the Difference of Two Squares Formula

The equation $a^2+b^2=c^2$ looks more friendly when written as $a^2=c^2-b^2$, because on the right side we see a difference of two squares: an old friend! Now if we write the equation in factorized form as

(1) $\quad a^2=(c-b)\cdot(c+b)$

then our chances of success look brighter. Let us solve the equation in this form.

To make progress, let us arbitrarily put $c-b=1$ and explore what happens. The relation implies that $b,\ c$ are consecutive integers; and from (1) we get $a^2=c+b$. Since a^2 is a sum of two consecutive integers, it is an odd number. So if we take an odd square and express it as a sum of two consecutive integers, it ought to yield a Pythagorean triple. It does — and this is exactly our Method #1!

To put this idea into action, we select a number n and consider the odd square $(2n+1)^2=4n^2+4n+1$. We write it as a sum $b+c$ of two consecutive integers:

$$b=2n^2+2n,\qquad c=2n^2+2n+1.$$

These values correspond to the following identity:

$$(2n+1)^2=(2n^2+2n+1)^2-(2n^2+2n)^2,$$

and they yield the following PT:

$$(2n+1,\ 2n^2+2n,\ 2n^2+2n+1).$$

Here are some PTs generated using this method (you will see that they are all PPTs):

n	1	2	3
PPT	(3, 4, 5)	(5, 12, 13)	(7, 24, 25)
n	4	5	6
PPT	(9, 40, 41)	(11, 60, 61)	(13, 84, 85)

The PPTs generated by this scheme have the feature that the largest two entries are consecutive numbers.

You will naturally want to ask: What was the need to insist that $c-b=1$? None at all! We need not have imposed the condition. Let us examine what happens if we change it to $c-b=2$; this means that b and c differ by 2. Now we get:

$$a^2=2(c+b).$$

From this we see that a^2 is an even number; therefore a is even, and $a=2n$ for some number n, giving $\frac{1}{2}a^2 = 2n^2$. Does this yield a solution? Yes. To put the scheme into action, we select a number n and write $2n^2$ as a sum $b+c$ of two integers differing by 2; we get:

$$b=n^2-1, \quad c=n^2+1, \quad b+c=2n^2.$$

These values correspond to the following identity:

$$(2n)^2=(n^2+1)^2-(n^2-1)^2,$$

and they yield the following PT:

$$(2n, n^2-1, n^2+1).$$

Here are some PTs generated this way (starting with $n=2$ since $n=1$ yields $b=0$):

n	2	3	4
PPT	(4, 3, 5)	(6, 8, 10)	(8, 15, 17)

n	5	6
PPT	(10, 24, 26)	(12, 35, 37)

We see that when n is odd, the method yields PTs whose numbers are all even, so they are not PPTs. But if n is even we do get PPTs.

Observe what we have accomplished: simply by imposing the conditions $c-b=1$ and $c-b=2$, we obtained two distinct families of PTs. It seems reasonable to expect that by changing these conditions to $c-b=3$, $c-b=4$, and so on, we should be able to generate new families of PTs. But we leave the exploration to you. There is much to discover along the way, maybe some which will surprise us, and much to prove ....

Remarks

Methods #1 – #3 yield infinitely many Pythagorean triples, but these constitute only a small subset of the full family of PTs. Method #4 does seem to have the potential to yield the entire family, but we have left the details to you.

In Part II of this article we shall examine how to generate the entire family of PPTs in a systematic and unified manner.