feature

Analyzing games of chance with math . . .

Fair Game

What are the odds that the game of chance you are playing is fair? How does the organizer of a lottery make a profit? How often does a customer win at a casino? Kamala Mukunda analyzes games of chance and brings great expectations within reach of an introductory course to probability.

KAMALA MUKUNDA

Many things in life boil down to a game of chance, actually! Getting insurance, gambling, telling the girl you like that you like her, buying a lottery ticket, riding a motorcycle without a helmet . . . all these risky activities have two key things in common. They have an *outcome* which is usually in terms of winning or losing something. For example, in gambling one can win or lose money; or it may be peace of mind. Also, there is a *probability* associated with each outcome; this can be calculated precisely in some cases, and only be estimated in other cases. Relative frequencies are one way of understanding the meaning of probability: if an experiment is carried out a large number of times, then the relative frequency of the occurrence of an event (in the 'long run') may be described as the probability of that event. An interesting topic in probability is the understanding of 'games of chance', defined broadly. One more useful feature of such games is that they can be played over and over again, which gives the idea of a 'long-run average' (a useful concept as you will see).

Let's start with a very simple game played between two friends A and B, using a coin. They decide that A wins if the outcome is heads, and B wins if the outcome is tails. Then they both put 10 rupees each in the middle, and toss! If it lands heads, A gets to keep the 20 rupees, but if it lands tails, A loses her 10 rupees. If the game is played only once, one of the friends is certain to be disappointed and the other one elated! But here's a question: assuming they play long enough, is this game fair to both players? Clearly, yes. Suppose they toss two coins, and A wins if both are same, while B wins if both are different. Again, the game is perfectly fair, because the four outcomes HH, TT, HT and TT are equiprobable (each of the outcomes has probability $1/2 \times 1/2 = 1/4$).

In fair games, the probability of winning is the same for both players. A nice way to formalize this is to draw a simple table showing the outcomes for each player, with their associated probabilities. For A, the outcome X could be -10 or $+10$ as below.

X	-10	$+10$
P(X)	1/2	1/2
X.P(X)	-5	$+5$

The third row is the product of X and P(X), and when we add these values for all values of X, we get something called an Expected Value for X, or E(X). E(X) is the long-run average of a person's outcomes over many trials, or in this case, it is the average of A's winnings over many, many repetitions of the same game with B. As you can see in the example, E(X) is zero! Of course it will be the same for B. That's why this is a fair game, because although on any given game only one of them can win, in the long run, neither is expected to win more often than the other.

Another interpretation of E(X) is that each time A and B play, they each should 'expect' to win 0 rupees! Of course no one in her right mind will really expect the winnings to be 0—we know that either you will lose 10 rupees or you will win 10 rupees—but this is what 'expected' means in a probability course. (By similar reasoning, each time you roll a die, you expect to get 3.5, even though you cannot possibly ever get it!)

In an actual run of 100 games, it may happen that A wins 54 times and B wins 46 times; or A wins 45 times and B wins 55 times; or A wins 60 times and B wins 40 times; etc. So A may leave the place with some winnings from B's pocket, or it may happen the other way round. That is part of what is meant by 'expected value'; we expect both A and B to win 50 games each, but we also expect some reasonable variation from that scenario. In this article, I will not go into the meaning of 'reasonable variation' (though it's certainly important and interesting), because a great deal of fun can be had with expected values alone.

Here is an example of an unfair game: A die is rolled, and A wins if the outcome if less than or equal to 4, whereas B wins if the outcome is greater than 4. The table below for B's winnings shows why:

X	-10	$+10$
P(X)	4/6	2/6
X.P(X)	$-20/3$	$+10/3$

On every game, B expects to win $-20/3 + 10/3 = -10/3 = -3.33$ rupees, or in other words he stands to lose each time.

You can see that the key in these tables is to be able to figure out the probability values, P(X). Tossing a coin and rolling a die seem to yield straightforward calculations . . . but you can quickly make it more difficult. Consider this game: as before, A and B place 10 rupees each on the table. A rolls a single die and if she gets a 3 or a 6, she wins and the game ends. If she does not, B rolls two dice and if he gets a 6 on either die he wins. If he gets no 6s, the game is drawn, and they each take their money back. (See Figure 1.)

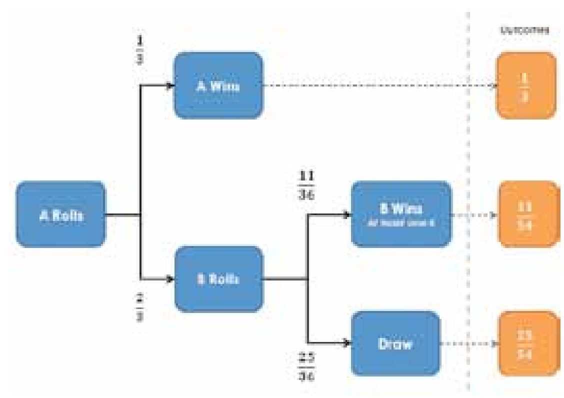

Fig. 1

The table of outcomes for B is below:

X	−10	+10	0
P(X)	1/3	11/54	25/54
X.P(X)	−10/3	+55/27	0

For B, E(X) = (−10/3 + 55/27) = −1.29 rupees, so B can expect to lose Rs. 1.29 every time he plays. If they played 100 times, he could expect to lose around 129 rupees, give or take a little. You can make a similar table of outcomes for A, using the probabilities computed above and you will see where the 129 rupees is going! To make this a fair game, we could say that for B to win, he needs to roll a single die and get an even number (probability 1/2). Figure 2 shows the tree diagram for this game; make the tables of outcomes for A and B to convince yourself that this is now a fair game.

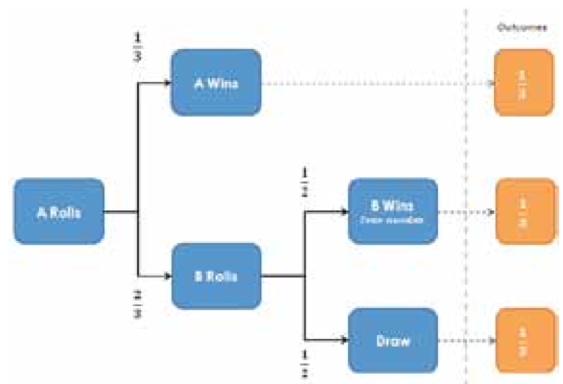

Fig. 2

A similar game was popular among French noblemen in the 1600s, the assumption being that it was a fair game, because it was thought that the probabilities were equal. They did not know about tree diagrams then, or how to calculate probabilities. It was when they noticed that whoever played in B's position would come out the loser in the long run that they wrote to the mathematician Blaise Pascal, who along with his friend Pierre de Fermat solved the problem and invented probability theory in the process. (I've included the problem at the end of this article for you to solve.)

Now it's time to introduce a third friend, C. He joins the fair game A and B are playing in Figure 2, and says to them, "If neither of you wins, I'll take the 20 rupees!" The outcomes have now changed, as you can see below. In the tree diagram, you need to replace 'Draw' with 'C wins'.

For A and B, E(X) = (−20/3 + 10/3) = −3.33:

X	−10	+10
P(X)	2/3	1/3
X.P(X)	−20/3	+10/3

For C, E(X) = 0 + 20/3 = 6.67:

X	0	+20
P(X)	2/3	1/3
X.P(X)	0	+20/3

So this seems like an unfair game, one that no A and B in their right minds would agree to play. But it is *exactly* the kind of game you are agreeing to play when you walk into a casino! (That's what C stands for, by the way.)

Of course, the numbers will not be quite so obviously tilted in favour of C. Instead, you may place 10 rupees on the table, and stand to win 100. If the probabilities are going to be as in the table below, then neither the casino nor you are gaining anything in the long run:

X	−10	+90
P(X)	90%	10%
X.P(X)	−9.00	+9.00

So they have to create a game only slightly different from the above, where the probabilities are more like this for you:

X	−10	+90
P(X)	91%	9%
X.P(X)	−9.10	+8.10

This ensures that you still feel like playing, but in the long run you're quite sure to leave a loser!Your E(X) of −1 rupee for each time you play may seem ridiculously small winnings for the casino, but if many, many people play this game many, many times, then it translates to big winnings for the casino (this is their long-run average working for them). *Remember, this includes the fact that some people will win, "just by chance"*. Ten thousand games a day makes 10,000 rupees for them, and with the profits they can easily afford some fine furniture, flashing lights and free food to make you return for more games!

It's relatively easy to create a game where the probabilities turn out as in the above table. The trick is to make players feel like they really have a chance to win, and one way is to charge relatively little to play, and dangle a sufficiently large win under your nose. Lotteries are like that.
A thousand people buy a one rupee ticket each, and the winner wins, not a thousand rupees but 900 rupees. The remaining 100 is for the organizers of the lottery. If I don't exactly know how many tickets are sold and therefore cannot calculate my probabilities, the ratio 1:900 may persuade me to buy a ticket. My table of outcomes:

X	-1	$+899$
P(X)	0.999	.001
X.P(X)	-0.999	$+0.899$

$E(X) = -0.1$.

Sometimes people feel if they buy ten tickets their chances are better. Let's see:

X	-10	890
P(X)	0.99	0.01
X.P(X)	-9.90	$+8.90$

$E(X) = -1.00$! Why did this happen? Look at it this way, if you bought all 1000 tickets, you'd be sure to win the 900 rupees, and your E(X) would be -100! There's always a casino in the background.

A quick look at Insurance Polices

We take a quick look at insurance policies and why they work. By now the language of tables is easy for you to follow, and I will use it in an oversimplified example of a one-year accident insurance for an individual, freely estimating probabilities and costs! In the table, X refers to the money the customer has at the end of a year. It may be negative (the premium paid to the insurance company) or positive (the company paid all costs incurred due to the accident). Let's assume the premium is 1000 rupees, and accident costs come to 100,000 rupees.

X	-1000 (the premium)	$100,000 - 1000 = 99,000$
P(X)	(as high as) 0.999	(as low as) 0.001
X.P(X)	-999	99

$E(X) = -900$, which means that every year you buy insurance you can expect to lose 900 rupees. You may wonder why anyone would buy an insurance policy under these conditions. To better understand this, you should play around with the table changing the cost and the probability of an accident (make both numbers larger). You will see that the expected value quickly jumps to a positive number. In other words, someone who estimates the probability of having an accident as relatively high, and who also imagines the prospect of a large bill for that accident, is more likely to take out the insurance policy, because they calculate their E(X) to be positive!

X	-1000	$200,000 - 1000 = 199,000$
P(X)	0.99	0.01
X.P(X)	-990	1990

However, where is the profit for the insurance company going to come from, if each customer's E(X) is positive? The trick is to adjust the premium so that the E(X) is a small negative number for each customer. Insurance companies don't rely on imagination and 'gut feel' to set premiums. Analysis of large sets of data on accident statistics and costs help insurance companies set premiums that ensure they make a profit, while at the same time making the customers feel that it is a good deal for them. For the company, the table of outcomes looks something like this with each customer.

X	$-$ accident costs (say a lakh of rupees)	$+$ premium amount
P(X)	(as low as) 0.01	(as high as) 0.99
X.P(X)	-1000	$+$ nearly the premium amount

E(X) just has to be a positive number; remember they have many, many customers, so thanks to long-run averages E(X) for each customer can be small. Therefore in this simplified example, the company can set premium at just over 1000 to make a decent profit and still attract customers.

Many of our daily decisions are in fact the result of mental 'calculations' from tables of outcomes. Depending on our personality, we assign different probabilities to different outcome values, and make our estimates of E(X). Without access to the kind of actuarial data that an insurance company

has, our estimates could easily be misleading!
I show my students who have just obtained a two-wheeler license the following table of outcomes for driving without a helmet. And I ask, is this a fair game, would you like to play it?

X	−Life	Discomfort of wearing a helmet
P(X)	(as low as) .001	(as high as) 0.999
X.P(X)	− depends how you value your life	Discomfort of wearing a helmet

Problems

1. The game that inspired Pascal and Fermat to invent probability theory involved rolling a single die four times, or rolling two dice 24 times. If you chose the former, you won on getting a six. If you chose the latter, you won on getting a double six. Which option should you choose? The prevailing assumption was that these two were equally likely to give a win, but their reasoning was wrong. They thought that since the chance of a six on each roll is 1/6, on four rolls the chance will be 4/6. They also reasoned that since the chance of a double six is 1/36, in 24 throws it will be 24/36, or 4/6 again, and therefore the game is fair. Though their calculations are wrong, the difference from 50–50 is so slight that it was only upon playing many, many times that it could be detected. In fact the game is not fair. Can you do the calculations correctly?

ANS: The probability of a six with four throws of a single die is $1-(5/6)^4$ which exceeds 50% (by just a small bit), and the probability of a double six with 24 throws of two dice is $1-(35/36)^{24}$ which is less than 50% (by just a small bit)!
2. A popular game in casinos and statistics textbooks is *roulette*. It consists of a spinning disc with 38 pockets and a ball that can fall into any one with equal probability. The pockets are numbered 0, 00 and 1 to 36. The 0 and 00 pockets are green, half the rest are black and half red (see the image, taken from http://www.kanzen.com/genimg/american-roulette-wheel-0-00-abb.jpg).


American style double zero roulette wheel.
Photo courtesy www.abbiati.it

Fig. 3

The wheel is spun, and before the ball lands in a pocket, you can place one of several bets.

- Bet on red (or black), odd (or even; crucially, 0 and 00 are not considered even numbers in roulette!), a number from 1–18 (or from 19–36). The winnings for all these kinds of bets are 'double your money'—you get back what you paid, plus an equal amount as winnings. Of course if you lose you forfeit whatever you staked.
- Bet on a single number; winning gives you what you staked, plus 35 times that amount.
- Bet on any two consecutive numbers (you win if the ball lands in either); winning gives you what you staked, plus 17 times that amount.
- Bet on any four consecutive numbers (you win if the ball lands in any of them); winning gives you what you staked, plus 8 times that amount.

Make the table of outcomes for a player (or for the casino) for each of these bets, calculating the E(X) values. Having done this it will be clear to you why the bets are arranged as they are. It will also be clear why there are two pockets numbered 0 and 00!

Suppose you wanted to set up a casino and decided to be greedy, offering lower winnings for each bet. For example, you could say that betting on a single number gives 20 times (instead of 35 times) the staked amount as winnings. As casino manager, your E(X) would shoot up for that particular bet. But why might you *not* do that?

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KAMALA MUKUNDA received a PhD in Educational Psychology from Syracuse University in 1992 and has been at Centre For Learning Bangalore since 1995 teaching various things including psychology, mathematics and statistics. She has written a book in 2009 titled 'What did you ask at school today?'. She may be contacted at kamala.mukunda@gmail.com