feature

Axioms of Paper Folding

How does an entertaining pastime such as paper folding evolve into a field of geometry? Can there be axioms about making creases on paper? Shiv Gaur talks about the axioms of the ancient but still richly-evolving field of origami, viewed as a 'cousin' of geometry, and then demonstrates its surprising power.

SHIV GAUR

The axiomatic system originated in ancient Greece. Axioms are "self-evident" truths which do not need proof; they serve as the starting points and building blocks upon which a deductive system is based–like **Euclidean geometry**. Some choices are available in formulating the axioms, but they should certainly be *consistent* as well as *independent* of one another; and it is desirable that they be *simple* ('user friendly') and *small in number*.

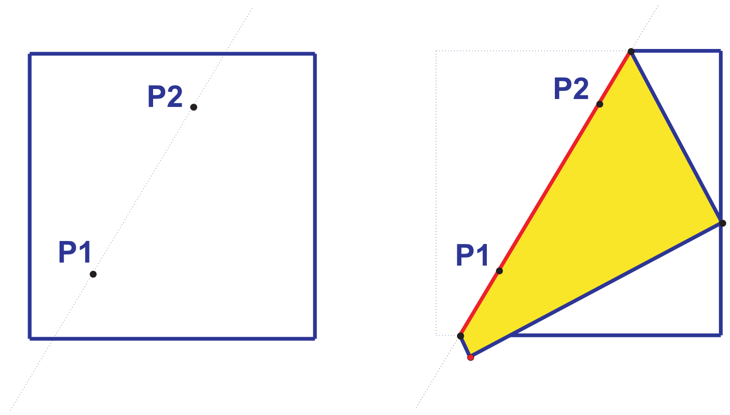
In origami, lines are replaced by *creases.* The question arises whether in paper folding too we can have a set of axioms which govern and tell us what various combinations of points and lines permit us to do.

In 1992 Humiaki Huzita formulated six operations, wherein a single crease could be created by aligning one or more combinations of points and lines on a sheet of paper; these came to be known as *Huzita Axioms*. In 2002 a Japanese origamist

Koshiro Hatori found a single-fold alignment that could not be described in terms of the six axioms, and this became the basis of the seventh axiom. The seven axioms have become known as the **Huzita-Hatori** axioms. Physicist, engineer and origamist Robert Lang proved that these axioms are *complete*, i.e., there can be no other way of defining a single fold in origami using alignments of points and lines.

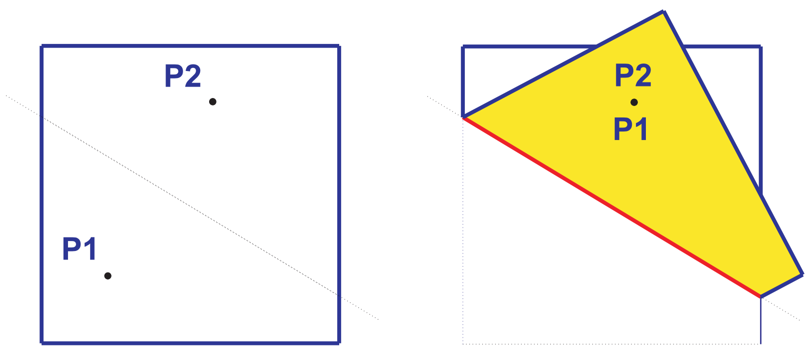
Axiom #01

Given two points P1 and P2, a line can be folded passing through both P1 and P2.



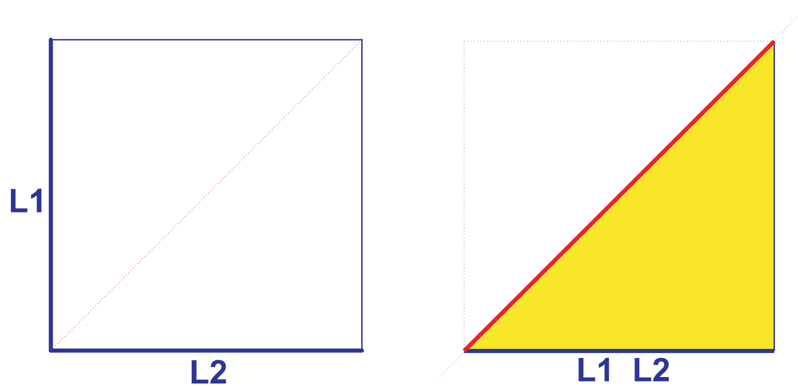
Axiom #02

Given two points P1 and P2, a line can be folded placing P1 onto P2.



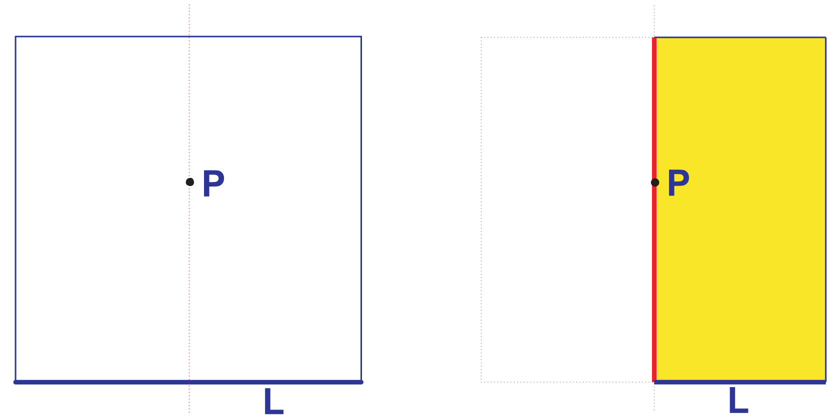
Axiom #03

Given two lines L1 and L2, a line can be folded placing L1 onto L2.



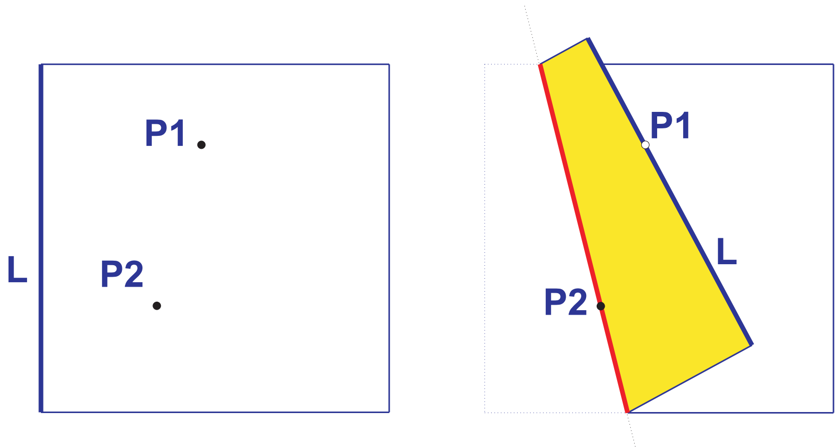
Axiom #04

Given a point P and a line L, a line can be folded passing through P, perpendicular to L.



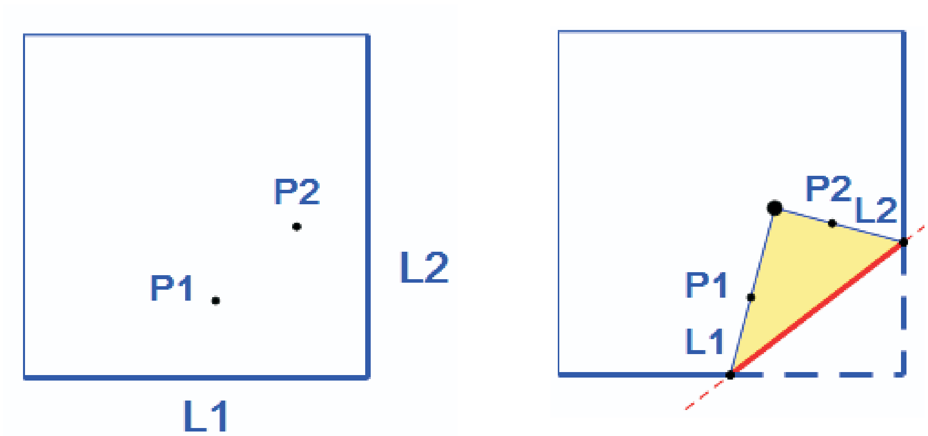
Axiom #05

Given two points P1 and P2 and a line L, a line can be folded placing P1 onto L and passing through P2.



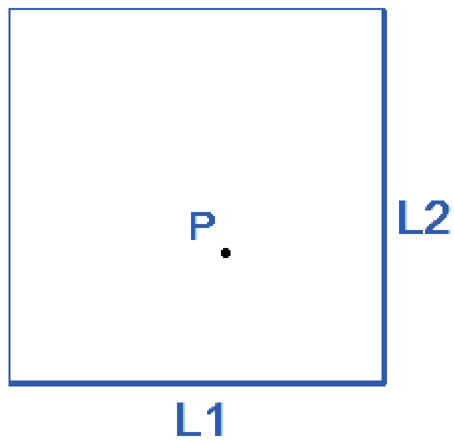
Axiom #06

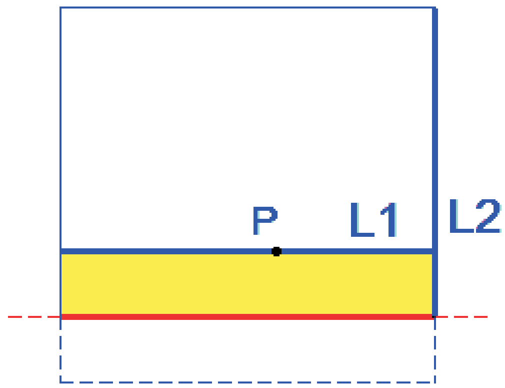
Given two points P1 and P2 and two lines L1 and L2, a line can be folded placing P1 onto L1 and placing P2 onto L2.



Axiom #07

Given a point P and two lines L1 and L2, a line can be folded placing P onto L1 and perpendicular to L2.

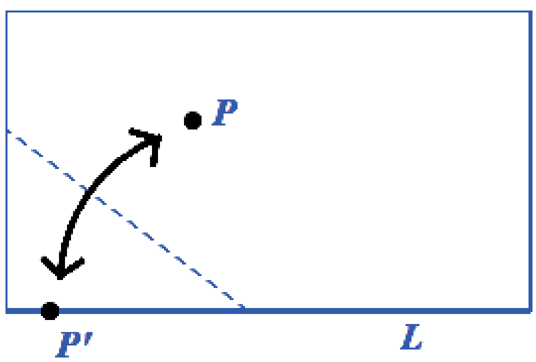




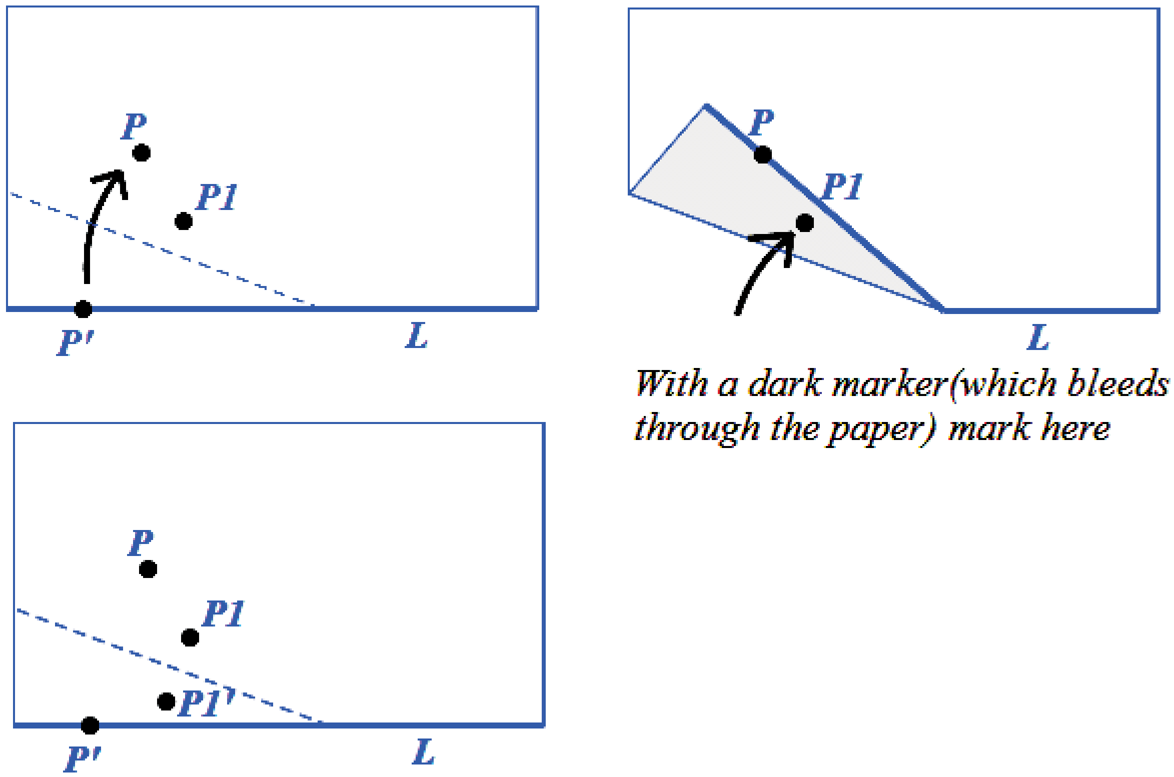
Axiom 6 is essentially about drawing the lines which simultaneously touch two parabolas; as such, it involves the solution of a *cubic equation*. Since a cubic equation can have up to three real roots, there can be up to three such common tangent lines. This means that Axiom 6 allows us to solve cubic equations! Such a possibility does not exist in regular Euclidean geometry, in which the instruments at our disposal—straightedge and compass—permit us to draw only straight lines and circles, and these never give rise to cubic equations. It follows that origami geometry is 'stronger' than Euclidean geometry! In particular, one can solve problems such as angle trisection and doubling of the cube, which are impossible using straight edge & compass.

As a small exercise, by actual folding of paper or by simulating with any dynamic geometry software, try experimenting and folding all the possibilities present in Axioms 5 and 6. Specifically one can try these:

1. On a sheet of paper consider one side as L. Mark a point P anywhere on the paper and keep folding L to P and crease the paper along L. Do this repeatedly (choose different points on L each time you map P to L). What shape emerges from doing this?



2. Next, repeat this experiment by drawing a circle on the paper, which now represents L, and repeat the procedure with P first inside and later outside the circle. What do you find?
3. Again, on a sheet of paper consider one side as L. Mark a point P anywhere on the paper. Take a second point P_1 anywhere. Fold P to L and using the resulting crease mark the reflection of point P_1. Choosing different points on line L repeat this procedure again and again. What curve emerges from the marked points?

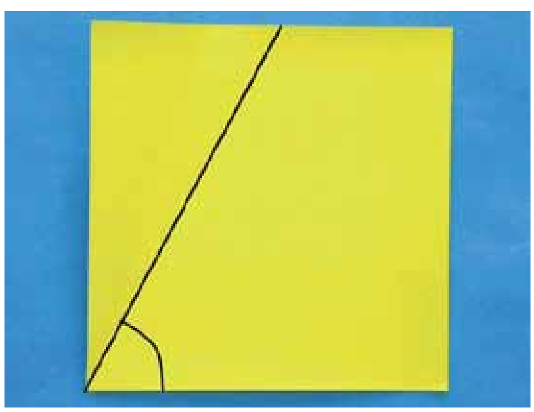


Any dynamic geometry software such as GeoGebra or Geometer's Sketchpad can simulate the above mentioned activities with the help of the 'Trace' function.

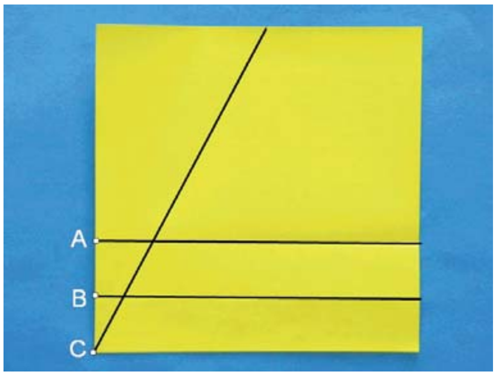
Trisecting an Angle

Now let's have a look at the steps involved in the trisection of an angle using the above mentioned axioms. This method is by H. Abe (from "Trisection of angle by H. Abe" (in Japanese) by K. Fusimi, in Science of Origami, a supplement to Saiensu (the Japanese version of Scientific American), Oct. 1980).

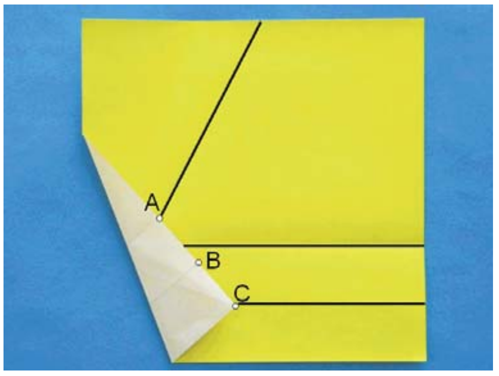
- Step 1: Fold a crease on a square paper creating an angle at the corner (to be trisected).



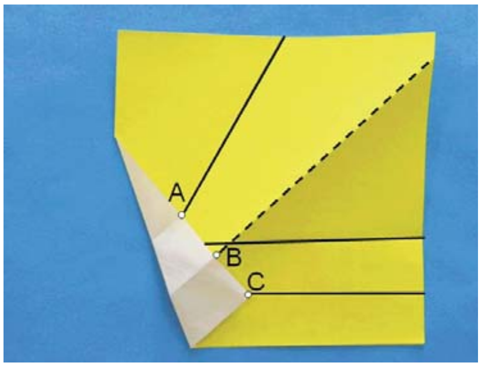
- Step 2: Make two horizontal creases at A and B by folding twice, so that AB = BC. Here C lies at the corner of the sheet.



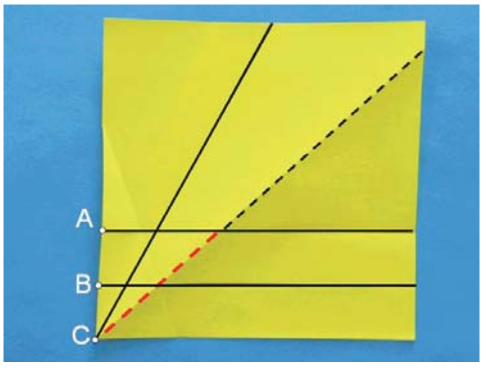
- Step 3: Fold point A onto the crease of the given angle, and point C onto the bottom crease.



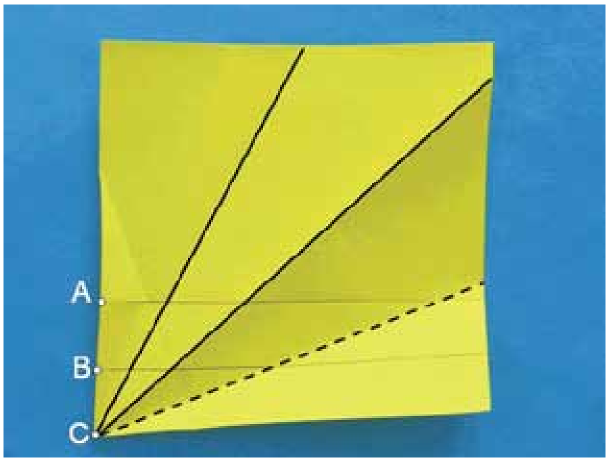
- Step 4: Extend the pre-creased line from B towards the top right corner (dashed line).



- Step 5: Unfold; extend the crease (dashed line) to the bottom left corner. This is one of the trisectors!



- Step 6: Folding the base to the trisector just found yields the other trisector of the angle.



We have thus drawn the two trisectors. (We leave the proof to you!) Such a construction is not possible in Euclidean geometry.

References

[1] http://en.wikipedia.org/wiki/Huzita%E2%80%93 Hatori_axioms
[2] http://kahuna.merrimack.edu/~thull/omfiles/ geoconst.html

A B.Ed. and MBA degree holder, Shiv Gaur worked in the corporate sector for 5 years and then took up teaching at the Sahyadri School (KFI). He has been teaching Math for 12 years, and is currently teaching the IGCSE and IB Math curriculum at Pathways World School, Aravali (Gurgaon). He is deeply interested in the use of technology (Dynamic Geometry Software, Computer Algebra) for teaching Math. His article "Origami and Mathematics" was published in the book "Ideas for the Classroom" in 2007 by East West Books (Madras) Pvt. Ltd. He was a guest speaker at IIT Bombay, TIME 2009. Shiv is an amateur magician and a Modular Origami enthusiast. He may be contacted at shivgaur@gmail.com.