feature

The Fine Art of Euclidean Geometry

Viviani's Theorem . . . And A Cousin

"I think it is said that Gauss had ten different proofs for the law of quadratic reciprocity. Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalise in different directions — they are not just repetitions of each other." Sir Michael Atiyah, interview in European Mathematical Society Newsletter, September 2004.

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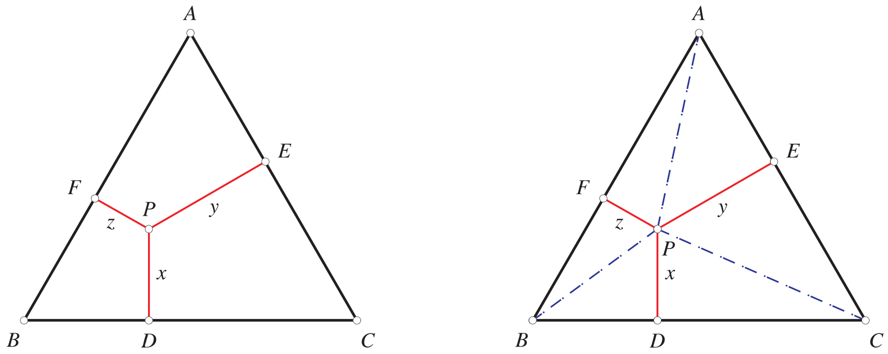
Viviani's theorem is one of those beautiful results of elementary geometry that can be found experimentally even by young children. It is appropriate from a pedagogic point of view, because it allows us to illustrate what it means to 'do mathematics': it gives us the opportunity to find a proof, and the opportunity to experience the pleasure of generalization.

The theorem is named after Vincenzo Viviani (1622–1703), an Italian mathematician-scientist and a disciple of Galileo during the last few years of his (Galileo's) life. (Readers may recall that Galileo was under 'house arrest' for the last several years of his life. Viviani was with him during part of that period, and helped in the compilation of the important book, *Discourses and Mathematical Demonstrations Relating to Two New Sciences*.)



(i) $x+y+z=$ constant (ii) Does the picture suggest a proof?

Fig. 1 Statement of Viviani's theorem, and a possible proof

Here is the statement of the theorem. Let $\triangle ABC$ be equilateral, and let P be any point in its interior (Figure 1 (i)). Then: *The sum of the distances from P to the sides of the triangle is a constant.* Thus, if perpendiculars PD, PE, PF are drawn from P to the sides BC, CA, AB, and their lengths are x, y, z, respectively, then the sum $PD+PE+PF=x+y+z$ is the same for all positions of P.

A way of proving the theorem is indicated in Figure 1 (ii). We draw segments PA, PB and PC (shown dashed). All we need to do is compute the area of $\triangle ABC$ in two different ways and study the resulting expressions; the constancy of $x+y+z$ follows. We discover as well what the 'constant value' is: *it equals the altitude of the triangle*. (Which makes sense: think of the different positions that P can occupy.) We invite you to complete the proof.

A Geometric Proof

A cardinal rule in the teaching-learning of mathematics is not to be satisfied with a single solution or proof — however pretty it is, or however satisfying! In that spirit we look for other proofs of Viviani's theorem. As a part motivator for this, we note that the proof suggested above, based on computation of area, is 'algebraic'. At a crucial step we factorize an expression and divide by one term, exploiting the fact that the three sides have equal length. For this reason we describe the proof as *essentially algebraic*, and we ask: *Is there an essentially geometric proof?* We now present such a proof.

In Figure 2 we have drawn a segment B_1C_1 through P, parallel to side BC (with B_1 on side AB, and C_1 on side AC). It is clear that for all positions of P on B_1C_1, the distance of P from BC remains the same. Let us now show that the sum of the distances from P to sides CA and AB is the same for all positions of P on B_1C_1. Accordingly, consider another point P_1 on B_1C_1, and drop perpendiculars P_1E_1 and P_1F_1 from P_1 on AC and AB. *We must show that $PE+PF=P_1E_1+P_1F_1$.* Drop perpendiculars $PQ \perp P_1F_1$ and $P_1R \perp PE$. In moving from P to P_1, the distance to side AB has increased by P_1Q, while the distance to side AC has decreased by PR. *So we must show that $P_1Q=PR$.*

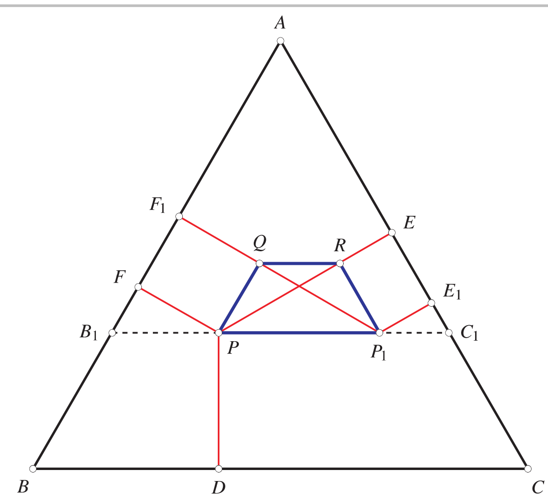


Fig. 2 Proof with a geometric flavour

But this follows readily. Since $PQ \perp P_1Q$ and $P_1R \perp PR$, points P, Q, R, P_1 are concyclic. Further, PR and P_1Q both subtend angles of 60° at the circumference of the circle. Hence they have equal length. (Note that $P\,Q\,R\,P_1$ is actually an isosceles trapezium.)

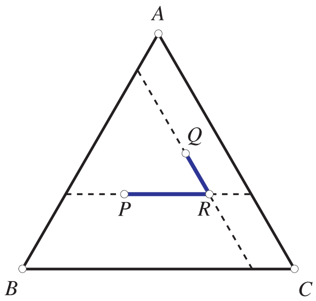


Fig. 3 Moving from P to Q using movements parallel to the sides

It follows from this that if P moves along a path parallel to any side, its sum of distances to the sides remains constant. Since one can move from any point within the triangle to any other point through movements parallel to the sides (in fact, just two such movements are needed; Figure 3 shows how to move from one interior point P to another one Q via the intermediate point R, using the path coloured blue), it follows that each point within the triangle has the same distance sum.

A Proof Using Vectors

It is possible to devise a proof using vectors based on the following principle. Let PB be a segment and let ℓ be a line passing through P (see Figure 4). Let the projection of PB on ℓ be PD. Then $PD = \overrightarrow{PB} \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector along the line ℓ (oriented suitably).

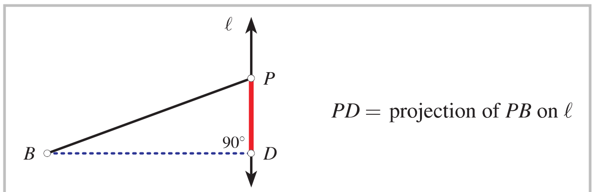


Fig. 4 Length as a projection

We apply this to the proof of Viviani's theorem. Let P be a point within the equilateral $\triangle ABC$. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be unit vectors perpendicular respectively to the sides BC, CA and AB, and oriented from the centre O of the triangle towards the midpoints of the sides (see Figure 5). From symmetry considerations we see that $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is zero. Making use of the 'projection principle' noted above, we infer that

$$PD = \overrightarrow{PB} \cdot \mathbf{u}, \quad PE = \overrightarrow{PC} \cdot \mathbf{v}, \quad PF = \overrightarrow{PA} \cdot \mathbf{w}.$$

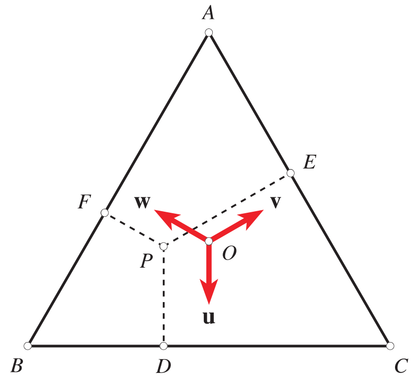


Fig. 5

Hence we must show that $\overrightarrow{PB} \cdot \mathbf{u} + \overrightarrow{PC} \cdot \mathbf{v} + \overrightarrow{PA} \cdot \mathbf{w}$ has a constant value, independent of P. Let Q be a second point within the triangle. The distance sum associated with Q is then $\overrightarrow{QB} \cdot \mathbf{u} + \overrightarrow{QC} \cdot \mathbf{v} + \overrightarrow{QA} \cdot \mathbf{w}$. Hence the difference between the two distance sums is

$$\begin{aligned}
&(\overrightarrow{PB} \cdot \mathbf{u} + \overrightarrow{PC} \cdot \mathbf{v} + \overrightarrow{PA} \cdot \mathbf{w}) \\
&\qquad - (\overrightarrow{QB} \cdot \mathbf{u} + \overrightarrow{QC} \cdot \mathbf{v} + \overrightarrow{QA} \cdot \mathbf{w}) \\
&\quad = (\overrightarrow{PB} - \overrightarrow{QB}) \cdot \mathbf{u} + (\overrightarrow{PC} - \overrightarrow{QC}) \cdot \mathbf{v} \\
&\qquad + (\overrightarrow{PA} - \overrightarrow{QA}) \cdot \mathbf{w} \\
&\quad = \overrightarrow{PQ} \cdot \mathbf{u} + \overrightarrow{PQ} \cdot \mathbf{v} + \overrightarrow{PQ} \cdot \mathbf{w} \\
&\quad = \overrightarrow{PQ} \cdot (\mathbf{u}+\mathbf{v}+\mathbf{w}) = 0,
\end{aligned}$$

since $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is zero. Hence the difference between the distance sums is 0, which means that the distance sums associated with P and Q are the same. Since this relation is true for any two points P and Q, it follows that the distance sum is a constant.

A Cousin of Viviani's Theorem

The clinching condition in the above proof is the fact that $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is the zero vector. This simple observation allows us to find another theorem which looks much like Viviani's theorem but is different from it. We shall call it a 'cousin' of Viviani's theorem.

What the vector proof shows is that if P is a point within an equilateral $\triangle ABC$, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are *any* three fixed unit vectors such that $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is the zero vector, then the sum of lengths of the projections of PA, PB, PC on $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ (respectively) will be a constant for all such points P. (The reason for this claim should be clear.)

But we get infinitely many theorems from this statement, because we can choose the three unit vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in infinitely many ways! All we need to ensure is that their (vector) sum is zero. Here is one possibility: *Let* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *be unit vectors along the directions* $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ *respectively*. It is obvious that the sum of these three unit vectors is zero (see Figure 6). This gives rise to the following theorem.

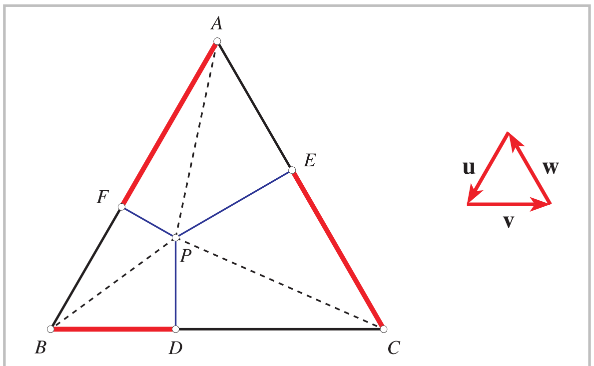


Fig. 6 A new theorem, somewhat like Viviani's theorem

Theorem. *Let $\triangle ABC$ be equilateral, and let P be a point in its interior. Let perpendiculars PD, PE, PF be dropped to BC, CA, AB respectively. Then $BD + CE + AF$ has the same value for all positions of P.*

It is easy to deduce what the constant value of $BD + CE + AF$ must be. Let P lie at the circumcentre of $\triangle ABC$; then $BD = CE = AF = a/2$ where a is the side of the triangle. Hence an equivalent claim is: $BD + CE + AF = 3a/2$ *for all positions of P*.

This yields yet another form of the claim! For, if $BD + CE + AF = 3a/2$, then we also have $CD + BF + AE = 3a/2$. So: $BD + CE + AF = CD + BF + AE$ *for all positions of P*. We give an alternate proof of this claim, based on the Pythagorean theorem.

Since $PB^2 = PD^2 + BD^2$ and similarly for PC^2 and PA^2, we have: $PB^2 - PC^2 = BD^2 - CD^2 = (BD - CD) \cdot (BD + CD) = a(BD - CD)$.

We have similar relations for $PC^2 - PA^2$ and $PA^2 - PB^2$. Hence:

$$\begin{cases} PB^2 - PC^2 = a(BD - CD), \\ PC^2 - PA^2 = a(CE - EA), \\ PA^2 - PB^2 = a(AF - FB). \end{cases}$$

The sum of the three quantities on the left is 0, and so therefore is the sum of the quantities on the right. It follows that

$$(BD - CD) + (CE - EA) + (AF - FB) = 0,$$

and hence that

$$BD + CE + AF = CD + BF + AE.$$

It follows that

$$BD + CE + AF = 3a/2 = CD + BF + AE.$$

Questions to Ponder

In closing we leave the reader some questions to ponder.

Q1: Viviani's theorem requires that P be a point within the equilateral $\triangle ABC$. Can a modification be found in the statement of the theorem which will make it applicable to points outside the triangle?

Q2: Can there be an 'inequality form' of Viviani's theorem (presumably, for triangles which are not equilateral)?

Q3: What generalization can be made of Viviani's theorem to polygons with a larger number of sides? Is there a class of polygons with the property that the sum of the distances from an interior point to the sides of the polygon is the same for every point? (It seems highly plausible that the property will be true for any regular polygon. Could it extend to polygons which are not regular?)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