Complete Family of Pythagorean Triples

From Random to Systematic Generation

feature

In Part I of this article we presented a few methods for generating Primitive Pythagorean Triples (PPTs). You will recall that they were all 'piece meal' in character. Now we present two more approaches which offer complete solutions to the PPT problem. Both are based on straightforward reasoning and simple algebra. And no PPT is left out: we capture the complete family in each case.

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At the start we recall the definition: a Pythagorean triple is a triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$. The triple is called 'primitive' if a, b, c have no common divisor exceeding 1; we call such a triple a 'Primitive Pythagorean Triple' (PPT for short). For example, $(5, 12, 13)$ is a PPT, while $(6, 8, 10)$ is a Pythagorean triple which is not a PPT.

Remark. We make the following number theoretic observation about PTs which are not PPTs. *If two numbers in a PT share a common factor exceeding* 1*, this factor divides the third number as well*. For example, $(9, 12, 15)$ is a PT, and its numbers 12 and 15 share the factor 3; this factor divides 9 as well. To see why this claim of divisibility will always be true, suppose that in the PT (a, b, c), both b and c are divisible by some integer k. Then k^2 divides both b^2 and c^2, hence k^2 divides a^2, since $c^2 - b^2 = a^2$; hence k divides a as well. This logic works no matter which two of a, b, c are divisible by a common factor. Hence, to check that a

PT is a PPT, it is enough to pick any two of its entries and check that they are coprime; the nice thing is that it does not matter which two entries we pick!

Generating the Full Family of PPTs By Solving Equations

Let (a, b, c) represent a PPT. We write its defining relation $a^2 + b^2 = c^2$ in the form

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1. \tag{1}$$

Let $u = a/c$ and $v = b/c$. Then u and v are positive rational numbers, and they have the same denominator (because no 'cancellation' can take place in either of the two fractions). Also, they lie between 0 and 1, and they satisfy the equation $u^2 + v^2 = 1$.

To solve this equation we transpose the terms and write it in the form $u^2 = 1 - v^2$. In this form it immediately looks more familiar, because we are able to make use of the well known 'difference of two squares' factor formula. Write the equation $u^2 = 1 - v^2$ as

$$u \cdot u = (1 - v) \cdot (1 + v), \quad \therefore \frac{u}{1 - v} = \frac{1 + v}{u}. \tag{2}$$

Denote the common value of $u/(1 - v)$ and $(1 + v)/u$ by t (in terms of the original quantities a, b, c we have $t = a/(c - b)$; note that t is a positive rational number, for it is the ratio of two positive rational numbers):

$$\frac{u}{1 - v} = t, \quad \frac{1 + v}{u} = t. \tag{3}$$

By cross-multiplication and transposing terms, we obtain a pair of simultaneous equations in u and v:

$$\begin{cases} u + tv &= t, \\ tu - v &= 1. \end{cases} \tag{4}$$

Treating t as a fixed quantity, we solve for u and v in the usual way (we do not give the steps here; please check the answer we have given); we obtain:

$$u = \frac{2t}{t^2 + 1}, \quad v = \frac{t^2 - 1}{t^2 + 1}. \tag{5}$$

Recall that t is a positive rational number. Let $t = m/n$ where m and n are positive, coprime integers. Since $u = a/c$ and $v = b/c$ we get, by substitution:

$$\frac{a}{c} = \frac{2 \cdot m/n}{(m^2/n^2) + 1} = \frac{2m\,n}{m^2 + n^2},$$

$$\frac{b}{c} = \frac{(m^2/n^2) - 1}{(m^2/n^2) + 1} = \frac{m^2 - n^2}{m^2 + n^2}.$$

Hence:

$$a : b : c = 2m\,n : m^2 - n^2 : m^2 + n^2. \tag{6}$$

It is easy to verify that if a, b, c satisfy these ratios then they satisfy the Pythagorean relation, because of the identity $(2m\,n)^2 + (m^2 - n^2)^2 = (m^2 + n^2)^2$. So: $(2m\,n,\ m^2 - n^2,\ m^2 + n^2)$ *is a PT for every pair of coprime integers* m, n *with* $m > n$.

Note that we only said 'PT', not 'PPT' — it could happen that the triple is a PT but not a PPT. Here are some examples of both kinds:

- $(m, n) = (8, 3)$ yields the triple $(48, 55, 73)$ which is a PPT.
- $(m, n) = (7, 3)$ yields the triple $(42, 40, 58)$ which is not a PPT as all its numbers are even. But note that we can recover a PPT from it by dividing all the numbers by their gcd which happens to be 2; we get the PPT $(21, 20, 29)$.
- $(m, n) = (5, 3)$ yields the triple $(30, 16, 34)$ which is not a PPT but yields the PPT $(15, 8, 17)$ on division by 2.

So $(m, n) = (8, 3)$ yields a PPT whereas $(m, n) = (7, 3)$ or $(5, 3)$ do not. If you experiment with various coprime pairs (m, n), and we urge you to do so, you will find that *you get a PPT precisely when m and n have opposite parity* (i.e., when one of them is odd, and the other one even; this may be expressed compactly by writing: $m + n$ *is odd*). Please experiment on your own and confirm this finding.

How do we prove this? The condition is clearly needed; for, if m, n have the same parity (which means in our context that they are both odd, as they are supposed to be coprime and so cannot both be even), then $2m\,n$, $m^2 - n^2$ and $m^2 + n^2$ will all be even numbers.

We now prove that if m and n are coprime and have opposite parity, then $2m\,n$, $m^2 - n^2$ and $m^2 + n^2$ are coprime. For this, it is enough if we show that $m^2 - n^2$ and $m^2 + n^2$ are coprime. (Recall the remark made at the start to see why.) Let k denote the gcd of $m^2 - n^2$ and $m^2 + n^2$. We present the proof that $k = 1$ as follows.

- Since m and n are coprime, so too are m^2 and n^2.
- Since k divides both the numbers $m^2 - n^2$ and $m^2 + n^2$, it divides their sum (which is $2m^2$) as well as their difference (which is $2n^2$); so k divides both $2m^2$ and $2n^2$.
- Since m and n have opposite parity, m^2 and n^2 have opposite parity. Hence $m^2 + n^2$ and $m^2 - n^2$ are odd, and k, being their gcd, is odd.
- Since k divides $2m^2$ and $2n^2$, and k is odd, it must be that k divides both m^2 and n^2.
- But m^2 and n^2 are coprime. Hence $k = 1$.

Thus $m^2 - n^2$ and $m^2 + n^2$ are coprime, as claimed, and the PT is a PPT. We conclude: *If m, n are positive coprime integers of opposite parity, and*

$$a = 2m\,n, \quad b = m^2 - n^2, \quad c = m^2 + n^2, \tag{7}$$

then (a, b, c) is a PPT. Table 1 lists some PPTs along with their (m, n) pairs.

A stronger claim

We can make a stronger statement: *The above scheme generates every possible PPT (a, b, c) in which a is even and b, c are odd.* Let us show why.

Let (a, b, c) be a PPT in which a is even, and b, c are odd. Let the fraction $t = a/(c - b)$ be written in its simplest form as m/n (so m, n are coprime). Working as shown above, we find that $a : b : c = 2m\,n : m^2 - n^2 : m^2 + n^2$. We now show that m, n have opposite parity. Suppose that m, n are both odd (obviously, they cannot both be even). Then $2m\,n$ and $m^2 + n^2$ are both of the form $2 \times$ an odd number, whereas $m^2 - n^2$ is a multiple of 4. Dividing through by 2 we find that it is b rather than a which is an even number. However we had supposed that a is even and not b. Hence it cannot be that m, n are both odd. So they must have opposite parity. But if m, n are coprime and have opposite parity, then $2m\,n$, $m^2 - n^2$ and $m^2 + n^2$ are coprime; we had shown this earlier. Now from the equalities $a : b : c = 2m\,n : m^2 - n^2 : m^2 + n^2$ and the fact that a, b, c are coprime as well as $2m\,n, m^2 - n^2, m^2 + n^2$, we can conclude that $(a, b, c) = (2m\,n, m^2 - n^2, m^2 + n^2)$, as required.

Example: Consider the PPT $(a, b, c) = (48, 55, 73)$. Here $t = a/(c - b) = 48/18 = 8/3$; so we take $m = 8$ and $n = 3$. Now check that $(m, n) = (8, 3)$ generates the PPT (48, 55, 73).

A number theoretic approach

To round off this discussion we shall derive the formula (7) in a completely different way, number theoretic in flavour. The key principle we use is the following proposition.

Proposition. *If r and s are coprime positive integers such that rs is a perfect square, then both r and s are perfect squares.*

For example, the product of the coprime numbers 4 and 9 is a perfect square, and each of these numbers is a perfect square. We invite you to prove the proposition.

Let (a, b, c) be a PPT in which a is even (and therefore both b and c are odd). From the relation $a^2 + b^2 = c^2$ we get $a^2 = c^2 - b^2 = (c + b)(c - b)$. We write this relation as follows:

$$\left(\frac{a}{2}\right)^2 = \frac{c+b}{2} \cdot \frac{c-b}{2}. \tag{8}$$

Since a, $c + b$ and $c - b$ are even numbers, the quantities $\frac{1}{2}a$, $\frac{1}{2}(c + b)$ and $\frac{1}{2}(c - b)$ are integers. We claim that $\frac{1}{2}(c + b)$ and $\frac{1}{2}(c - b)$ are coprime. To see why, suppose that d is a common divisor of $\frac{1}{2}(c + b)$ and $\frac{1}{2}(c - b)$; then d must divide their sum $(= c)$ as well their difference $(= b)$. Hence d divides c as well as b. But we know that b and c are coprime. Hence $d = 1$, and $\frac{1}{2}(c + b)$ and $\frac{1}{2}(c - b)$ too are coprime.

From (8) we see that the product of the coprime numbers $\frac{1}{2}(c + b)$ and $\frac{1}{2}(c - b)$ is a perfect square. Hence each of them is a perfect square! Let $\frac{1}{2}(c + b) = m^2$ and $\frac{1}{2}(c - b) = n^2$. By addition and subtraction we get $c = m^2 + n^2$ and $b = m^2 - n^2$. From (8) we get $a = 2m\,n$. Hence there exist coprime integers m and n such that $(a, b, c) = (2m\,n, m^2 - n^2, m^2 + n^2)$.

We illustrate this step with an example. Take the PPT $(a, b, c) = (48, 55, 73)$ in which a is even, and b and c are odd, as required. For this PPT we have: $\frac{1}{2}(c + b) = \frac{1}{2}(73 + 55) = 64$ and $\frac{1}{2}(c - b) = \frac{1}{2}(73 - 55) = 9$. Observe that $\frac{1}{2}(c + b)$

Table 1. A list of some (m, n) pairs and the PPTs they yield.

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$
2	1	3	4	5
3	2	12	5	13
4	1	8	15	17
4	3	24	7	25
5	2	20	21	29
5	4	40	9	41
6	1	12	35	37
6	5	60	11	61
7	2	28	45	53
7	4	56	33	65
7	6	84	13	85
8	1	16	63	65
8	3	48	55	73
8	5	80	39	89
8	7	112	15	113
9	2	36	77	85
9	4	72	65	97
9	8	144	17	145
10	1	20	99	101
10	3	60	91	109
10	7	140	51	149
10	9	180	19	181
11	2	44	117	125
11	4	88	105	137
11	6	132	85	157
11	8	176	57	185
11	10	220	21	221
12	1	24	143	145
12	5	120	119	169
12	7	168	95	193
12	11	264	23	265

and $\frac{1}{2}(c-b)$ are perfect squares. Hence $m = \sqrt{64} = 8$ and $n = \sqrt{9} = 3$. Please check that by using these values of m, n in (7) we get the same PPT with which we started, (48, 55, 73).

It remains to show that m, n have opposite parity. But we leave the task to you.

Remark. The approaches we have presented above are only two of many different ways of tackling the Pythagorean equation. Here are some other directions we could have taken: (i) the double angle formulas of trigonometry, (ii) complex numbers, (iii) coordinate geometry and quadratic equations. The reassuring thing is that all these different approaches give exactly the same general result. We shall come back to some of these approaches later.