Ramanujan Numbers

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In the last issue of **At Right Angles** we had noted an observation that Ramanujan had made about the number 1729: *It is the least positive integer that can be written as the sum of two positive cubes in more than one way* (namely, as $10^3 + 9^3$ and as $12^3 + 1^3$), and we asked you to find the next integer, after 1729, with the same property.

A 'brute force' computer assisted search reveals the following such numbers:

$$\begin{aligned}
1729 &= 9^3 + 10^3 &&= 1^3 + 12^3, \\
4104 &= 9^3 + 15^3 &&= 2^3 + 16^3, \\
20683 &= 19^3 + 24^3 &&= 10^3 + 27^3, \\
39312 &= 15^3 + 33^3 &&= 2^3 + 34^3, \\
40033 &= 16^3 + 33^3 &&= 9^3 + 34^3, \\
64232 &= 26^3 + 36^3 &&= 17^3 + 39^3.
\end{aligned}$$

We can list more such equalities by scaling: $13832 = 18^3 + 20^3 = 2^3 + 24^3$ (from the entry for 1729). But we regard these as uninteresting and do not list them. The numbers with the desired property are seen to be: 1729, 4104, 20683, 39312, 40033, 64232, ... The next Ramanujan number after 1729 is thus 4104.

Ramanujan's Solution

Instead of a brute force method, can we not look for approaches that are more worthy of being called 'mathematical'?

When we have an equation and we must find integers satisfying it, the equation is referred to as a Diophantine equation (after the Greek mathematician Diophantus). Two well known examples: (i) the Pythagorean equation $a^2 = b^2 + c^2$, which gives rise to Pythagorean triples; (ii) the Fermat equation $a^n = b^n + c^n$ (with $n > 2$). For taxicab numbers the defining equation is $a^3 + b^3 = c^3 + d^3$.

It turns out that it is possible to solve the equation $a^3 + b^3 = c^3 + d^3$ in a systematic way. The great eighteenth century mathematician Euler did so. So did Srinivasa Ramanujan, during the period when he was still in India, composing his now-famous notebooks. (This was before he went to England, in 1914, at the invitation of G H Hardy.) Here are the formulas he found: if u and v are arbitrary integers, positive or negative, and

$$a = 3u^2 + 5uv - 5v^2, \quad b = 4u^2 - 4uv + 6v^2,$$

$$c = 5u^2 - 5uv - 3v^2, \quad d = 6u^2 - 4uv + 4v^2,$$

then $a^3 + b^3 + c^3 = d^3$, identically. This is nearly the same as our equation, except that c has come on the 'wrong' side. Clearly, this can be fixed by a simple change of sign.

For example, if we put $u = 1$ and $v = -2$ we get $a = -27$, $b = 36$, $c = 3$, $d = 30$, hence:

$$(-27)^3 + 36^3 + 3^3 = 30^3.$$

Since each term in this equality is divisible by 3 we may divide it out without losing anything; we get $(-9)^3 + 12^3 + 1^3 = 10^3$, and therefore by exchanging terms:

$$12^3 + 1^3 = 9^3 + 10^3.$$

Now we see why this identity came so readily to Ramanujan when Hardy mentioned the number 1729; he had found it out many years earlier!

Other (u, v) combinations yield more such nice and non-obvious relations:

- from $(u = 1, v = 2)$, we get $7^3 + 14^3 + 17^3 = 20^3$;
- from $(u = 1, v = -3)$, we get $7^3 + 54^3 + 57^3 = 70^3$;
- from $(u = 2, v = 3)$, we get $3^3 + 36^3 + 37^3 = 46^3$;
- from $(u = 2, v = -3)$, we get $23^3 + 94^3 = 63^3 + 84^3$, and this yields yet one more Ramanujan number: 842751.

It is difficult to say how Ramanujan found these formulas. But that complaint holds for just about everything that Ramanujan found!

Readers who wish to see Euler's derivation of the general integral solution of the equation $a^3 + b^3 + c^3 = d^3$ should consult the book by G H Hardy and E M Wright, *Introduction to the Theory of Numbers*.