Fun Problems

1. A Magic Triangle

We are familiar with the notion of a magic square. Here we consider a related notion: that of a *magic triangle*. In Figure 1 we see a triangle with six circles on its three sides.

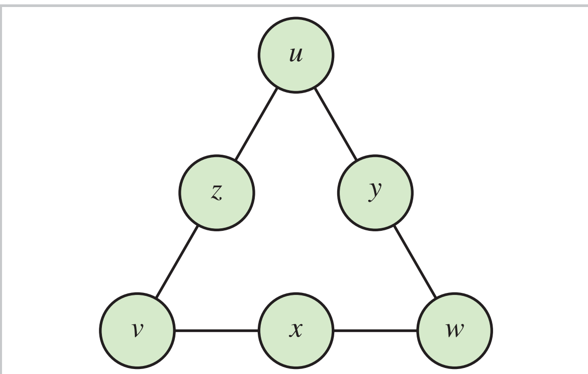

Fig. 1

Using the digit set $\{1, 2, 3, 4, 5, 6\}$ we must put one digit into each circle, using up all the six digits, in such a way that the sum of the numbers on each side is the same. That is, (u, v, w, x, y, z) should be a permutation of $(1, 2, 3, 4, 5, 6)$, and the quantities $u + z + v$, $v + x + w$ and $w + y + u$ must all be the same. Hence the name, 'magic triangle'. We ask: *In how many different ways can this be done?* We consider two arrangements to be the 'same' if one can be obtained from the other by flipping it over along some axis, or by rotating it about the centre of the triangle through some angle. (Strictly speaking, we should say 'congruent' rather than 'same'.) This may be achieved by agreeing to orient the triangle so that $u < v < w$ (i.e., looking only at the vertices, the smallest vertex number is at the top, and the largest vertex number is at bottom right).

We answer the question of 'how many ways' and in the process we uncover some pretty relationships which we shall describe as "theorems about magic triangles". The first one is: *In a magic triangle as defined here, the sum of the numbers at the vertices is a multiple of 3. So is the sum of the numbers at the 'middles' of the three sides.*

Let $C = u + v + w$ be the sum of the three corner numbers, and let $M = x + y + z$ be the sum of the three middle numbers. (The claim made is then: *C and M are multiples of* 3.)

Now, clearly:

$$C + M = 21, \tag{1}$$

since between them, $\{u, v, w\}$ and $\{x, y, z\}$ exhaust all of the numbers $\{1, 2, 3, 4, 5, 6\}$, and the sum of these six numbers is 21. Let s denote the common sum of the three numbers on each side.

Then we have:

$$\begin{cases} u+z+v=s, \\ v+x+w=s, \\ w+y+u=s, \end{cases} \tag{2}$$

hence by addition, $2(u+v+w)+(x+y+z)=3s$, i.e.,

$$2C+M=3s. \tag{3}$$

From (3) and (1) we get, by subtraction:

$$\begin{cases} C=3s-21=3(s-7), \\ M=21-C=3(14-s), \end{cases} \tag{4}$$

implying that both C and M are multiples of 3, just as we had claimed.

We have proved our first theorem about magic triangles!

Again, since C and M are each sums of three distinct numbers from the set $\{1, 2, 3, 4, 5, 6\}$, each of them is at least $1+2+3=6$, and at most $4+5+6=15$. It follows that the pair (C, M) is one of the following: (6, 15), (9, 12), (12, 9), (15, 6). The corresponding values of s are 9, 10, 11, 12. Hence we have the following possibilities:

	Possibility #1	Possibility #2	Possibility #3	Possibility #4
C	6	9	12	15
M	15	12	9	6
s	9	10	11	12

It is clear that these can be the only possibilities. However this by itself does not imply that all these possibilities can be realized, that such configurations really do exist; it may happen that some other condition 'comes in the way'. We can find out only by trial and error. Doing so, we find that each combination listed can indeed be realized. For proof we simply give the magic triangles (Figure 2).

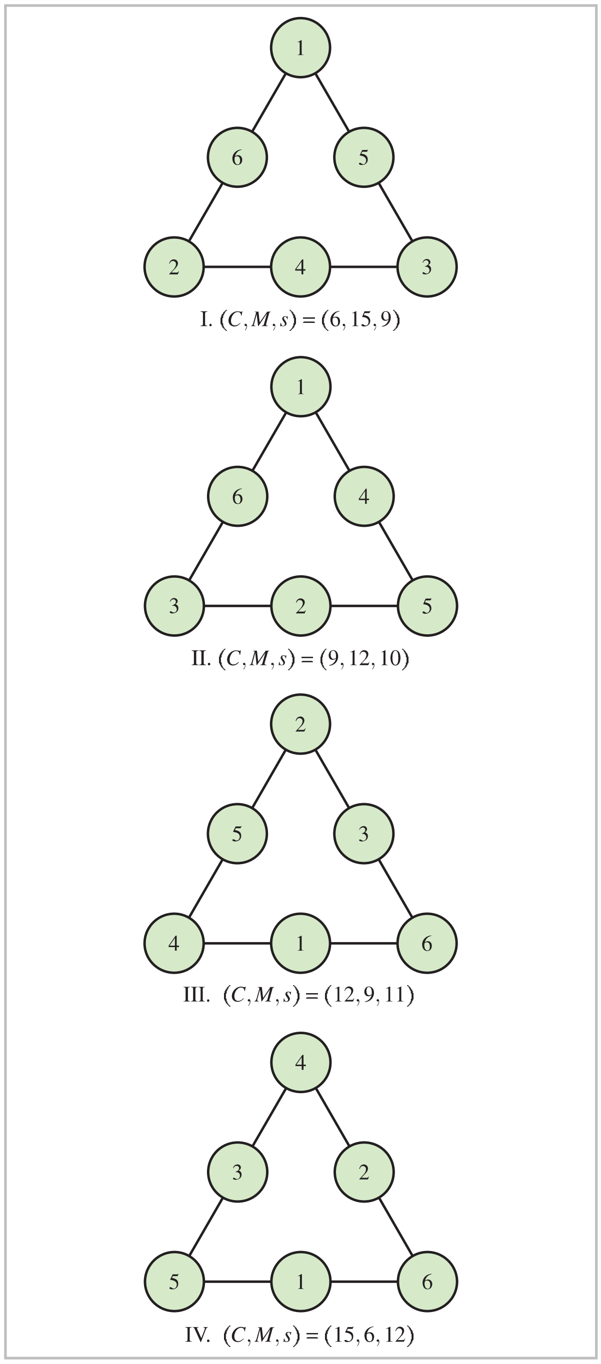

Fig. 2 The four possibilities

We see that *there are precisely four different magic triangles*. We have discovered our second theorem — and proved it at the same time.

Note the curious symmetry between triangles I and IV (each one is a sort of 'turned out' version of the other), and between triangles II and III. One could call it a kind of 'duality'.

On examining the configurations we find another theorem, which we had not anticipated: *In each magic triangle, the difference between the number at a vertex and the number at the middle of the opposite side is the same for all three vertices.* (The difference is 3 in configurations I and IV, and 1 in configurations II and III.)

Perhaps you will uncover more such theorems?

2. Problems for Solution

Notation. We introduced cryptarithms in the last issue of *At Right Angles*. We pose a few more problems of this genre here. But we make a change in the notation: *To denote a two-digit number with tens digit A and units digit B we use the notation* $\overline{AB}$ *and not AB.* The line covering AB serves to tell us that this is the intended meaning. Without the line we would not be able to distinguish between the two-digit number with tens digit A and units digit B and the ordinary product AB.

Problem I-2-F.1 Find an algebraic proof of the property stated above: *In a magic triangle, the difference between the number at a vertex and the number at the middle of the opposite side is the same for all three vertices.* That is, $u - x = v - y = w - z$ (with reference to Figure 1).

Problem I-2-F.2 Explore the analogous problem in which the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are placed along the sides of a triangle, one at each vertex and two on the interiors of each side, so that the sum of the numbers on each side is the same. What different theorems can be found for this configuration? Conduct a complete exploration.

Problem I-2-F.3 Show that the cryptarithm

$$\overline{AT} + \overline{RIGHT} = \overline{ANGLE}$$

has no solutions! (The problem has nine unknowns: $A, T, R, I, G, H, N, L, E$.)

Problem I-2-F.4 Solve the following cryptarithm:

$$\overline{CATS} \times 8 = \overline{DOGS}.$$

Problem I-2-F.5 Solve this cryptarithm:

$$\overline{ABCDEF} \times 5 = \overline{FABCDE}.$$

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3. Solutions of Problems from Issue-I-1

Problem I-1-F.1 *To solve the cryptarithm* $\overline{ABCD} \times 4 = \overline{DCBA}$.

- Let $x = \overline{ABCD}$ and $y = \overline{DCBA}$. Both x and y are four digit numbers. Since A is the leading digit of x we take A to be non-zero.
- Since $4x$ is a four digit number, $A = 1$ or 2. As A is the ones digit of $4x$, it must be even. Hence $A = 2$.
- Since D is the leading digit of $4x$, it follows that $D = 8$ or 9. Since the ones digit of $4x$ is 2 (and not 6), it follows that $D = 8$.
- Noting the 'carry' of 3 from the multiplication $D \times 4$, and the fact that there is no 'carry' from the multiplication $B \times 4$, it follows that

  $$4(10B + C) + 3 = 10C + B,$$

  giving $2C = 13B + 1$. The relation implies that B is odd.
- Since B and C are digits, it must be that $B = 1$ and therefore $C = 7$.
- Hence $x = 2178$ and $y = 8712$. It is easily checked that $y = 4x$.
- So the given cryptarithm, $\overline{ABCD} \times 4 = \overline{DCBA}$ has *precisely one solution*.

Comment. After getting $A = 2$ and $D = 8$ there are other ways of getting B and C. We have only described one such way, and we encourage you to find other ways.

Problem I-1-F.2 *To solve the cryptarithm* $(\overline{TWO})^2 = \overline{THREE}$.

- Looking at their leading digits (both are T), we realize that $T = 1$.
- Since $\overline{TWO}^2 < 20000$, it follows that $\overline{TWO} < 142$.
- As the ones digit of a perfect square is one of 0, 1, 4, 5, 6, 9, it follows that E is one of these digits. Hence $\overline{EE}$ is one of the numbers 00, 11, 44, 55, 66, 99.

- Now we recall the test for divisibility by 4: *A number n is divisible by* 4 *if and only if the number formed by the last two digits of n is itself divisible by* 4*.* We can make a stronger statement. Let n' denote the number formed by the last two digits of n. Then: *n and* n' *leave the same remainder under division by* 4. (To see why, observe that all powers of 10 higher than 10^2 are multiples of 4.)
- Next, recalling that all squares leave remainder 0 or 1 under division by 4, we see that EE leaves remainder 0 or 1 under division by 4. Hence we eliminate the possibilities 11, 55, 66 and 99 for $\overline{EE}$. So $E = 0$ or 4.
- If $E = 0$ then $O = 0$ too (two different letters cannot represent the same digit), and we disallow this. Hence $E = 4$, and $O = 2$ or 8. So the number formed by the last two digits of $\overline{TWO}$ is either $10W + 2$ or $10W + 8$.
- Suppose that $O = 2$. Since $(10W + 2)^2 = 100W^2 + 40W + 4$, we get $W = 1$ or 6 (as the last two digits of $(\overline{TWO})^2$ are 44). But 6 is too large (remember that $\overline{TWO} < 142$), and 1 has been 'used up'. So $O \neq 2$.
- Hence $\overline{TWO} \in \{108, 128, 138\}$.
- Testing these three possibilities we find that only the last one 'works' and we get the answer: $138^2 = 19044$. Check it out for yourself!

Comment. After getting $O = 2$ or 8 there are other ways possible. As earlier, we encourage you to find these ways on your own.

Problem I-1-F.3 To count the total number of 1s in the string 1, 2, 3, ..., 99, 100, 101, ..., 999998, 999999.

The neatest way of solving this is to pad each number in the given string with a suitable numbers of 0s from the left side, so that every number in the string has six digits. We also start the list with 000000. So the list is:
000000, 000001, 000002, ..., 999998, 999999.

There are now 10^6 numbers in the list, each with six digits, so the total number of digits in the entire string is simply 6×10^6. The crucial observation to make is that the number of 1s, 2s, 3s, ..., 8s, 9s in the new string is exactly the same as in the original string; only the number of 0s has changed.

Now by symmetry it should be clear that each of the digits 0, 1, 2, ..., 9 occurs with the same frequency in the new string. Hence there are 6×10^5 occurrences of each of the ten digits in the new string.

So the number of 1s in the original string is $6 \times 10^5 = 60{,}000$. (The number of 2s, 3s, ..., 9s is exactly the same. Only the number of 0s is different.)