Problems for the Middle School

Problem Editor : R. ATHMARAMAN

1. Problems for Solution

Problem I-2-M.1 Using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 once each can you make a set of numbers which when added and subtracted in some order yields 100 as the answer?

(For example, you could make the collection {132, 58, 40, 69, 70} and try the 'sum' $132-58+40-69+70$. But that does not work!)

Problem I-2-M.2 Let all the natural numbers be listed, except the multiples of 3:

$$1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, \ldots.$$

Find a simple formula for the n-th term of the above sequence, in terms of n.

Problem I-2-M.3 Let all the natural numbers be listed, except the perfect squares:

$$2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, 19, \ldots.$$

Find a simple formula for the n-th term of the above sequence, in terms of n.

Problem I-2-M.4 Amar, Akbar and Antony are three friends. The average age of any two of them is the age of the third person. Show that the total of the three friends is divisible by 3.

Problem I-2-M.5 A set of consecutive natural numbers starting with 1 is written on a sheet of paper. One of the numbers is erased. The average of the remaining numbers is $5\frac{2}{9}$. What is the number erased?

Problem I-2-M.6 The average of a certain number of consecutive odd numbers is A. If the next odd number after the largest one is included in the list, then the average goes up to B. What is the value of $B-A$?

Problem I-2-M.7 101 marbles numbered from 1 to 101 are divided between two baskets A and B. The marble numbered 40 is in basket A. This marble is removed from basket A and put in basket B. The average of all the numbers of marbles in A increases by $\frac{1}{4}$; the average of all the numbers of marbles in B also increases by $\frac{1}{4}$. Find the number of marbles originally present in basket A. (From the 1999 Dutch Math Olympiad.)

2. Solutions to problems from Issue-I-1

Solution to problem I-1-M.1

(a) *Right triangles in which the longer leg and hypotenuse are consecutive natural numbers:* (3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41).

(b) *Right triangles in which the legs are consecutive natural numbers:* (3, 4, 5), (20, 21, 29).

(c) *Rectangles with integer sides and a diagonal 25:* Sides 7 and 24; sides 15 and 20.

(d) *Two PPTs free from prime numbers:* (16, 63, 65), (33, 56, 65).

Solution to problem I-1-M.2 *Let (a, b, c) be a PT. Is it possible that among a, b, c:*

(i) *All three are even?* Possible; e.g., (6, 8, 10).

(ii) *Exactly two of them are even?* Not possible; if two of a, b, c are even, the third one must be even.

(iii) *Exactly one of them is even?* Possible; e.g., (3, 4, 5).

(iv) *None of them is even?* Not possible; if two of a, b, c are odd, then the third one must be even, necessarily.

Solution to problem I-1-M.3 *Let (a, b, c) be a PT. Is it possible that among a, b, c:*

(i) *All three are multiples of 3?* Possible; e.g., (9, 12, 15).

(ii) *Exactly two of them are multiples of 3?* Not possible; if two of a, b, c are multiples of 3, the third one too must be a multiple of 3.

(iii) *Exactly one of them is a multiple of 3?* Possible; e.g., (3, 4, 5).

(iv) *None of them is a multiple of 3?* Not possible. For the proof please see the solution to Problem I-1-S-2 elsewhere in this issue, where we prove more.

Solution to problem I-1-M.4 *To list PPTs in which one of the numbers in the PPT is 60.*

So we must find all pairs (m, n) of coprime integers, with opposite parity, such that one of $m^2 - n^2$, $2m\,n$, $m^2 + n^2$ is 60.

Since m, n have opposite parity, $m^2 - n^2$ and $m^2 + n^2$ are odd; so we cannot have $m^2 - n^2 = 60$ or $m^2 + n^2 = 60$. The only possibility is $2m\,n = 60$, i.e., $m\,n = 30$. In what ways can we express 30 as a product of two coprime positive integers? (They will automatically be of opposite parity, since 30 is divisible by 2 but not by 4.) Here are the ways: 30×1, 15×2, 10×3, 6×5. Hence (m, n) can be (30, 1), (15, 2), (10, 3) or (6, 5). This yields four possible PPTS: (899, 60, 901), (221, 60, 229), (91, 60, 109), (11, 60, 61).

Solution to problem I-1-M.5 *Take two fractions with product 2. Add 2 to each one, and multiply each by the* LCM *of the denominators. You get two natural numbers which are the legs of an integer sided right triangle.*

Let the two fractions be a/b and $2b/a$ where a, b are coprime. Adding 2 to each we get $(a + 2b)/b$ and $2(a + b)/a$. Multiplying by the LCM of their denominators which is $a\,b$, we get $a(a + 2b)$ and $2b(a + b)$. To verify the property we must show that $a^2(a + 2b)^2 + (2b)^2(a + b)^2$ is a perfect square. You should verify that this expression simplifies to $(a^2 + 2a\,b + 2b^2)^2$.

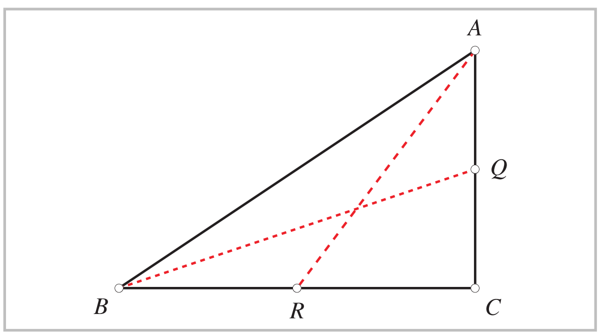

Fig. 1 Problem I-1-M-6

Solution to problem I-1-M.6 *The medians of a right triangle drawn from the vertices of the acute angles have lengths 5 and $\sqrt{40}$. What is the length of the hypotenuse?*

The situation is depicted in Figure 1, with $AR = 5$ and $BQ = \sqrt{40}$. Let the sides of $\triangle ABC$ be a, b, c.

Apply the Pythagorean theorem to $\triangle ARC$ and $\triangle BQC$:

$$AR^2 = b^2 + \left(\frac{a}{2}\right)^2, \quad BQ^2 = a^2 + \left(\frac{b}{2}\right)^2,$$

$$\therefore \quad AR^2 + BQ^2 = \frac{5\left(a^2+b^2\right)}{4},$$

so $5(a^2+b^2)/4 = 25+40 = 65$. Hence $a^2+b^2 = 52$. It follows that $c = \sqrt{52}$.

In general, if the medians drawn from the acute angles of a right triangle have lengths u and v, then the hypotenuse c is given by $5c^2 = 4(u^2+v^2)$.

Solution to problem I-1-M.7 *ABCD is a square of side 1; P and Q are the midpoints of sides AB and BC; PC and DQ meet at R. What can be said about $\triangle PRD$?*

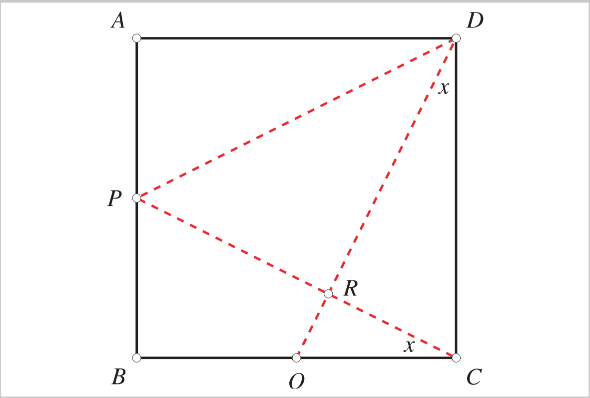

Fig. 2 Problem I-1-M.7

The situation is depicted in Figure 2. From $\triangle PBC \cong \triangle QCD$ it follows that the two angles marked x are equal, and so $\angle DRC$ is a right angle. So $\triangle PRD$ is a right triangle. To find its sides we use the principle of similarity. Observe that $\triangle CRQ \sim \triangle CBP$. Hence $CR/CQ = CB/CP$. Next, note that $CP^2 = 1^2 + (1/2)^2 = 5/4$, hence $CP = \sqrt{5}/2$. Also, $CQ = 1/2$. From this we get $CR/(1/2) = 1/(\sqrt{5}/2)$, hence $CR = 1/\sqrt{5}$. In the same way we get $DR = 2/\sqrt{5}$. We now get the length of PR:

$$PR = \frac{\sqrt{5}}{2} - \frac{1}{\sqrt{5}} = \frac{3}{2\sqrt{5}}.$$

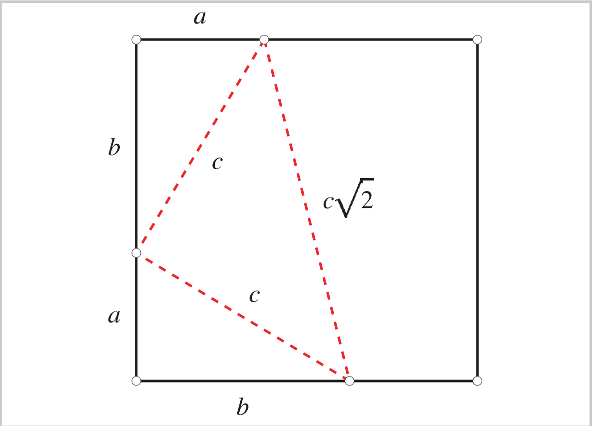

Fig. 3 Problem I-1-M.9

So the sides of the right triangle PRQ are in the following ratios:

$$PR : DR : PD = \frac{3}{2\sqrt{5}} : \frac{2}{\sqrt{5}} : \frac{\sqrt{5}}{2} = 3 : 4 : 5.$$

The right triangle PRD is a $3:4:5$ triangle. A pleasant surprise!

Solution to problem I-1-M.8 *Find all right triangles with integer sides such that their perimeter and area are* numerically *equal.*

Note the phrase 'numerically equal'; *area can never equal perimeter*, as the two are dimensionally distinct. An example of a triangle satisfying the condition is the one with sides (6, 8, 10); its area is $(6 \times 8)/2 = 24$ square units, and its perimeter is $6+8+10 = 24$ units. We solve the problem in full in the 'Senior Problems' section.

Solution to problem I-1-M. 9 *If a, b are the legs of a right triangle, show that*

$$\sqrt{a^2+b^2} < a+b \le \sqrt{2(a^2+b^2)}.$$

For proof see Figure 3, in which $c = \sqrt{a^2+b^2}$. The relation $c < a+b$ follows from the triangle inequality ("Any two sides of a triangle are together greater than the third one"), and $a+b \le \sqrt{2(a^2+b^2)}$ follows from the fact that the least distance between a pair of opposite sides of the square is $a+b$, and $\sqrt{2(a^2+b^2)} = c\sqrt{2}$ is the length of *one* possible segment connecting the same pair of sides.