problem corner

Problems for the Senior School

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'Area equals Perimeter'

In the 'Problems for the Middle School' in Issue-I-1 we asked for a listing of all integer sided right triangles with the property that the area *numerically* equals the perimeter. We study this problem in some detail here. Let a,b,c be the sides of the triangle, c being the hypotenuse. Then we have the following relations:

$$\text{area} = \frac{a\,b}{2}, \quad \text{perimeter} = a+b+c, \quad c^2 = a^2+b^2.$$

Hence we have:

$$a+b+\sqrt{a^2+b^2} = \frac{ab}{2},$$

$$\therefore \sqrt{a^2+b^2} = \frac{ab}{2} - a - b.$$

From the last relation we get, by squaring:

$$a^2+b^2 = \left(\frac{a\,b}{2} - a - b\right)^2$$

$$= \frac{a^2b^2}{4} - a^2b - a\,b^2 + 2a\,b + a^2 + b^2.$$

Simplifying, we get the relation $a^2b^2 - 4a^2b - 4a\,b^2 + 8a\,b = 0$. The expression on the left has $a\,b$ as a factor, and as this is not zero, we get:

$$ab - 4a - 4b + 8 = 0.$$

We must find all pairs (a, b) of positive integers satisfying this equation. Note that we have two unknowns but one equation; so the equation may have many pairs of solutions. It is an example of a *indeterminate Diophantine equation*. To solve it we use factorization. The expression $ab - 4a - 4b + 8$ does not factorize but if the '8' were replaced by '16' we would get a nice factorization. And that gives us the idea for a solution:

$$ab - 4a - 4b + 8 = (ab - 4a - 4b + 16) - 8$$

$$= (a-4)(b-4) - 8.$$

Since $ab - 4a - 4b + 8 = 0$ we get

$$(a-4)(b-4) = 8.$$

Hence $\{a-4, b-4\}$ is one of the following: $\{8,1\}$, $\{4,2\}$. It follows that $(a,b) = (12,5)$ or $(8,6)$. *Hence there are just two integer sided right triangles with the property that the area and perimeter are numerically equal. These are the triangles with sides* $(5,12,13)$, *with area and perimeter numerically equal to* 30; *and* $(6,8,10)$, *with area and perimeter numerically equal to* 24.

Problems for Solution

Problem I-2-S.1 A series of numbers beginning with 2012 is in AP (arithmetic progression) as well as GP (geometric progression). Find the sum of the first 100 terms of the series.

Problem I-2-S.2 Find the sum of all four digit numbers with the following property: *the sum of the first two digits equals the sum of the last two digits*. Also compute the number of such numbers.

Problem I-2-S.3 Show that no term of the sequence 11, 111, 1111, 11111, . . . is the square of an integer.

Problem I-2-S.4 The radius r and the height h of a right-circular cone with closed base are both an integer number of centimetres, and the volume of the cone in cubic centimetres is equal to the total surface area of the cone in square centimetres. Find the values of r and h.

Problem I-2-S.5 Given a $\triangle ABC$ and a point O within it, lines AO, BO and CO are drawn intersecting the sides BC, CA and AB at points P, Q and R, respectively. Prove that

$$\frac{AR}{RB} + \frac{AQ}{QC} = \frac{AO}{OP}.$$

Problem I-2-S.6 The triangular numbers 1,3,6,10,15,21,28, . . . are numbers of the form $n(n+1)/2$ for positive integers n. The square numbers 1,4,9,16,25,36,49, . . . are numbers of the form n^2 for positive integers n. Show that every triangular number greater than 1 is the sum of a square number and two triangular numbers.

Solutions of Problems in Issue-I-1

The following facts are needed. Let n be any integer; then: n^2 leaves remainder 0 or 1 modulo 3; n^2 leaves remainder 0 or 1 modulo 4; n^2 leaves remainder 0, 1 or 4 modulo 5; and n^2 leaves remainder 0, 1 or 4 modulo 8.

Solution to problem I-1-S.1 *For a PPT (a,b,c), one of a,b is odd and the other is even, and the even number is a multiple of* 4.

Certainly, a and b cannot both be even. Suppose that both a and b are odd. Then $a^2 \equiv 1 \pmod 4$ and also $b^2 \equiv 1 \pmod 4$, which yields $a^2 + b^2 \equiv 2 \pmod 4$, i.e., $c^2 \equiv 2 \pmod 4$.

But no such square exists; all squares are $0 \pmod 4$ or $1 \pmod 4$. Hence it cannot happen that both a,b are odd. So one of $\{a,b\}$ is even, and the other one is odd.

Suppose that a is even and b is odd. Then c is odd, and we have $b^2 \equiv 1 \pmod 8$ and also $c^2 \equiv 1 \pmod 8$, implying that $c^2 - b^2$ is a multiple of 8. Hence a^2 is a multiple of 8. This implies that a is an even multiple of 2, i.e., a is a multiple of 4.

Solution to problem I-1-S.2 *For a PPT (a,b,c), the product abc is a multiple of* 60.

We have already shown that if (a,b,c) is a PPT then one of $\{a,b\}$ is a multiple of 4, and hence that ab is a multiple of 4. If we can show that one of $\{a,b,c\}$ is a multiple of 3, and one of $\{a,b,c\}$ is a multiple of 5, then the problem will be solved since abc will then be divisible by $3 \times 4 \times 5 = 60$.

Suppose that both a and b are non-multiples of 3. Then $a^2 \equiv 1 \pmod 3$ and also $b^2 \equiv 1 \pmod 3$, and so $a^2 + b^2 \equiv 2 \pmod 3$. Therefore $c^2 \equiv 2 \pmod 3$. But there is no square of this form.

Hence it cannot be that both of a,b are non-multiples of 3. So one of a,b is a multiple of 3.

Suppose that a,b,c are non-multiples of 5. Then a^2, b^2, c^2 are all 1 or 4 $\pmod 5$. But none of the possibilities 'fits'; it is impossible to have $a^2 + b^2 \equiv c^2 \pmod 5$ within these possibilities. Hence it cannot be that a,b,c are all non-multiples of 5. So one of a,b,c is a multiple of 5.

Hence $\{a,b,c\}$ contains a multiple of 3, a multiple of 4 and a multiple of 5. Since 3,4,5 are coprime, it follows that abc is a multiple of 60.

Solution to problem I-1-S.3 *Any right-angled triangle with integer sides is similar to one in the Cartesian plane whose hypotenuse is on the x-axis and whose three vertices have integer coordinates.*

Suppose the given triangle has integer sides a,b,c where c is the hypotenuse. We construct a triangle with sides ac,bc,c^2, the side of length c^2 having end-points $B'(0,0)$ and $A'(c^2,0)$. Since its sides bear the ratios $a : b : c$, it is similar to the given triangle and hence is a right triangle (Figure 1). Its height is $h = (a\,c \times b\,c)/c^2 = ab$. It is easily verified that the coordinates of the third vertex are integers; for, if D' is the foot of the perpendicular from C' to side $A'B'$, then $B'D' = \sqrt{a^2c^2 - a^2b^2} = a^2$ and $C'D' = b^2$.

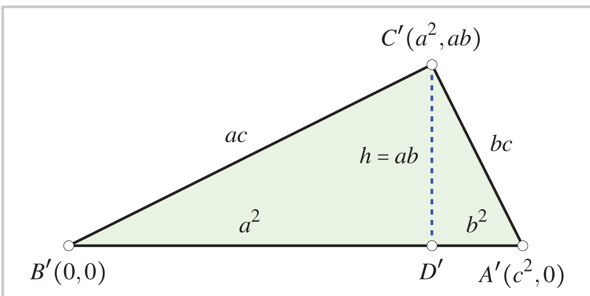

Fig. 1

Solution to problem I-1-S.4 *Let a,b and c be the sides of a right-angled triangle. Let θ be the smallest angle of this triangle. Show that if $1/a$, $1/b$ and $1/c$ too are the sides of a right-angled triangle, then $\sin\theta = (\sqrt{5} - 1)/2$.*

Proof. Suppose that θ lies opposite side a; then $a < b \le c$, hence $1/a > 1/b \ge 1/c$, and the hypotenuse of the triangle with sides $1/a$,$1/b$,$1/c$ is $1/a$, so $1/b^2 + 1/c^2 = 1/a^2$. Multiplying by a^2 and using $\sin\theta = a/c$, $\tan\theta = a/b$ we get:

$$\tan^2\theta + \sin^2\theta = 1, \quad \therefore\ \tan^2\theta = \cos^2\theta,$$

$$\therefore\ \sin^2\theta = \cos^4\theta$$

hence $(\sin\theta - \cos^2\theta)(\sin\theta + \cos^2\theta) = 0$. Since $\sin\theta + \cos^2\theta > 0$, the quantity $\sin\theta - \cos^2\theta$ must vanish. Hence $\sin^2\theta + \sin\theta - 1 = 0$, and $\sin\theta = (\sqrt{5} - 1)/2$.

Solution to problem I-1-S.5 *Find all Pythagorean triples (a,b,c) in which: (i) one of a,b,c equals 2011; (ii) one of a,b,c equals 2012.*

(i) Suppose $a = 2011$; then $c^2 - b^2 = 2011^2$, hence $(c - b)(c + b) = 2011^2$. Since 2011 is prime (did you know that?), the only way of accomplishing this is by setting $c - b = 1$ and $c + b = 2011^2$. Solving these we get $b = 2022060$ and $c = 2022061$. So the triple is (2011,2022060,2022061). Of course we can swap the first two entries. Suppose $c = 2011$; then $a^2 + b^2 = 2011^2$. To solve this we shall appeal to a result which we only state for now: *Let p be a prime number of the form* $3 \pmod 4$*. Suppose that x and y are integers such that $x^2 + y^2$ is a multiple of p. Then both x and y are multiples of p.*

The result may be easily verified for small p of the stated form, e.g., $p = 3, 7, 11$. To prove the result in full generality requires knowledge of the 'Little Theorem of Fermat'. The result applies since 2011 is a prime number of the form $3 \pmod 4$. It implies that both a and b are multiples of 2011. Let $a_1 = a/2011$ and $b_1 = b/2011$; then $a_1^2 + b_1^2 = 1$, which clearly has no solution in positive integers. Hence the equation $a^2 + b^2 = 2011^2$ too has no solution in positive integers.

So there are just two PTs in which one of the numbers is 2011: the one listed above and one of its permutations.

(ii) Suppose $a = 2012$; then $c^2 - b^2 = 2012^2$. By writing $2012^2 = 2^4 \times 503^2$ and noting that $c - b$ and $c + b$ have the same parity, we see that $(c - b, c + b)$ is one of the following: $(2, 2^3 \times 503^2)$, $(2^2, 2^2 \times 503^2)$, $(2^3, 2 \times 503^2)$. Solving three sets of equations we find that (a,b,c) can be any of the following:

(2012, 1012035, 1012037),

(2012, 506016, 506020),

(2012, 253005, 253013).

As earlier, we can also swap the first two entries.

Suppose $c = 2012$; then $a^2 + b^2 = 2^2 \times 503 \equiv 0 \pmod{503}$. As 503 is a prime number of the form $3 \pmod 4$ the result stated above applies; so both a and b are multiples of 503. Let $a_1 = a/503$ and $b_1 = b/503$; then $a_1^2 + b_1^2 = 2^2$, which clearly has no solution in positive integers. Hence the equation $a^2 + b^2 = 2012^2$ too has no solution in positive integers.

Solution to problem I-1-S.6 *Find all PPTs (a,b,c) in which a,b,c are in GP.*

Suppose (a,b,c) is a PPT in which a,b,c are in GP. Then $b^2 = ac$, and also $b^2 = c^2 - a^2$. Hence $c^2 - a^2 = ac$. Let $x = c/a$. Dividing $c^2 - a^2 = ac$ by a^2 we get $x^2 - 1 = x$, hence $x^2 - x - 1 = 0$, and $x = (\sqrt{5} + 1)/2$. This number is irrational, whereas c/a, a ratio of two positive integers, must be rational. So no such PPT exists.

Solution to problem I-1-S.7 *In any triangle, show that the sum of the squares of the medians equals $\frac{3}{4}$ of the sum of the squares of the sides.* See below for the solution.

Solution to problem I-1-S.8 *The figure shows a $\triangle ABC$ in which P, Q, R are points of trisection of the sides, with $BP = \frac{1}{3}BC,\ CQ = \frac{1}{3}CA,\ AR = \frac{1}{3}AB$. Show that the fraction $(AP^2 + BQ^2 + CR^2)/(BC^2 + CA^2 + AB^2)$ has the same value for every triangle.*

We solve a generalized version of this problem which includes Problem I-1-S-7. Let $\triangle ABC$ be given, and let P, Q, R be points on BC, CA, AB respectively such that with $BP/BC = CQ/CQ = AR/AB = 1/n$ where n is a given number (Figure 2). We wish to find the value of $(AP^2 + BQ^2 + CR^2)/(BC^2 + CA^2 + AB^2)$. Observe that $n = 2$ yields Problem I-1-S-7, and $n = 3$ yields Problem I-1-S-8.

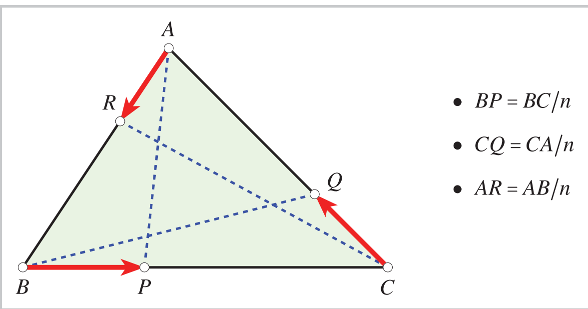

Fig. 2

We solve this using vectors. Let $\vec{u}, \vec{v}, \vec{w}$ represent the vectors $\vec{BP}, \vec{CQ}, \vec{AR}$ respectively. Then $\vec{BC} = n\vec{u}$, $\vec{CA} = n\vec{v}$, $\vec{AB} = n\vec{w}$. Since $\vec{BC} + \vec{CA} + \vec{AB} = 0$ (the zero vector), it follows that $n(\vec{u} + \vec{v} + \vec{w}) = 0$ and hence that $\vec{u} + \vec{v} + \vec{w} = 0$. By dotting each side of this relation with itself we get:

$$u^2 + v^2 + w^2 = -2(\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u}),$$

where $u^2 = \vec{u}\cdot\vec{u}$, etc. (Here u denotes the length of $\vec{u}$, etc.) Now $\vec{AP} = \vec{AB} + \vec{BP} = n\vec{w} + \vec{u}$, and similarly for $\vec{BQ}$ and $\vec{CR}$. Hence:

$$\begin{aligned} AP^2 + BQ^2 + CR^2 \\ &= (n\vec{w} + \vec{u})^2 + (n\vec{u} + \vec{v})^2 + (n\vec{v} + \vec{w})^2 \\ &= (n^2 + 1)(u^2 + v^2 + w^2) + 2n(\vec{u}\cdot\vec{v} \\ &\quad + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u}) \\ &= (n^2 - n + 1)(u^2 + v^2 + w^2), \end{aligned}$$

and since $BC^2 + CA^2 + AB^2 = n^2(u^2 + v^2 + w^2)$, we get:

$$\frac{AP^2 + BQ^2 + CR^2}{BC^2 + CA^2 + AB^2} = \frac{n^2 - n + 1}{n^2}.$$

This is the required answer. Note that the ratio does not depend on the shape of the triangle. It depends only on n.

If $n = 2$ (Problem I-1-S-7) the ratio simplifies to $3/4$; if $n = 3$ (Problem I-1-S-8) the ratio simplifies to $7/9$. So we have solved two problems in one stroke, and also obtained a more general result in the process.

As a side remark we note that the fraction $(n^2 - n + 1)/n^2$ achieves its *least value* of $3/4$ when $n = 2$. Try proving this for yourself.