Middle School Problems

Understanding Circular Motion

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Here are some problems related to circular motion, along with their solutions and some extension activities.

Problem VII-2-M.1: **On Coins and Plates**

A. Coin Rolling Outside

(i) A coin (or disc) rolls around another similar coin held down firmly, maintaining constant contact and without slipping. After it has executed one turn around the fixed coin, how many times has it turned around itself?

(ii) What if the fixed coin or disc was twice the size of the rolling coin/disc?

(iii) Or n times the size? (See Figure 1.)

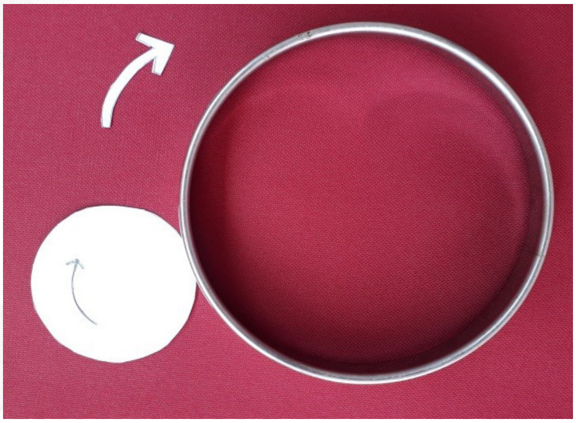

Figure 1.

Keywords: circular, rolling, angle, clock, orbit

B. Coin Rolling Inside

(iv) A coin/disc could also roll in a similar fashion along the edge of a circular depression (see Figure 2). Assuming the depression to be twice the size of the disc, if the disc executes one turn around the rim, how many times has it turned around itself?

(v) What if the depression were n times the size of the disc? (If the disc and depression were of the same size no movement would be possible.)

Figure 2.

Problem VII-2-M.2: **On Analog Clocks**

(i) How many times do the hour hand and minute hand of an analog clock/timepiece overlap in a 12 hour period? The two hands overlap at 12:00. What is the exact time when they do so again?

(ii) How many times do the two hands occupy diametrically opposite positions?

(iii) How many times do the two hands find themselves at right angles to each other

a. With hour hand 'downstream'

b. With minute hand 'downstream' (all in a 12 hour period)?

Problem VII-2-M.3: **On Planetary Motion**

Two planets circle the same sun in concentric circular orbits located in the same plane, with the sun at the centre, both moving in an anticlockwise direction. Planet A, occupying the inner orbit, takes x days to circle the sun once, while planet B, occupying the outer orbit, takes y days, with $y > x$. (Let us consider the 'days' to be 'earth days.')

(i) If they are in a straight line with the sun today, both being on the same side of the sun, after how many days will they again be in a line with the sun (again being on the same side of the sun)? Note that they need not occupy the same initial positions in their orbits.

(ii) Would the answer to the above be the same if the requirement is that they are in a line but on opposite sides of the sun on both occasions?

(iii) Or if they are to be at right angles with planet A 'downstream' on both occasions?

(iv) As above, but with planet B 'downstream'?

Solutions

Problem VII-2-M.1: **On Coins and Plates**

A. Coin rolling outside: When a circular disc rolls on a path, the number of turns it makes around itself equals the path length divided by the circumference of the disc. In the case of a disc (of radius R_{roll}) rolling around another stationary circular disc (of radius R_{stat}), this turns out to be the ratio of their circumferences or, equivalently, of their radii. However, in the process of making a circuit around the stationary disc, the moving disc executes one extra turn around itself. So the number of turns is given by the expression $\frac{R_{stat}}{R_{roll}} + 1$. Applying this formula, we get the following answers:

(i) Twice

(ii) Thrice

(iii) $n + 1$ times.

B. Coin rolling inside: The argument is similar to the one above. In this case the movement along the rim is in a sense/direction opposite to the movement around itself. Therefore the required expression is $\frac{\text{Rstat}}{\text{Rroll}} - 1$. Applying this formula, we get the answers (iv) Once (v) n – 1 times. Do try it out with actual materials.

Problem VII-2-M.2: **On Analog Clocks**

After 12:00, the minute hand races ahead of the hour hand, completing one full turn in 1 hour. By this time the hour hand has moved an angular distance of 30º. Now the minute hand needs to catch up with the hour hand. The time taken for this is obtained by dividing 30º by the angular velocity of the minute hand relative to the hour hand, which is 360º/hour -30º/hour = 330º/hour. The result is $\frac{1}{11}$ hour. So the exact time when the overlap happens is $1\frac{1}{11}$ hours past 12:00. Similar arguments show that subsequent overlaps happen at times $2\frac{2}{11}$ hours, $3\frac{3}{11}$ hours, etc. The last of the series is $11\frac{11}{11}$ hours which is actually 12 hours again. So the overlap happens 11 times in 12 hours at equally spaced intervals of time and positions on the dial.

Or one could reason like this: Let the angle between the hour hand and the 12 'o'clock position be x and the angle between the minute hand and the 12 'o'clock position be y. Now we have $y = 12x - 360N$ degrees, where N is an integer with $0 \le N \le 11$. When the hands overlap, we have $x = y = 12x - 360N$, which leads to $x = \frac{360N}{11}$. If $N = 0$, $x = y = 0$, which stands for noon/midnight. As N takes values 1, 2, 3,……10, x takes values $\frac{360}{11}, \frac{360 \times 2}{11}, \frac{360 \times 3}{11}, \ldots\ldots \frac{360 \times 10}{11}$.

If $N = 11$, then $x = \frac{360 \times 11}{11}$ which again stands for noon/midnight. So there are 11 overlapping situations equally spaced in time and position on the dial, as can be practically demonstrated with an analog clock/timepiece.

The answer to all sub-questions (ii) and (iii) is the same as above: 11 times in 12 hours. That is, any specified relative position of the hands occurs on 11 equally spaced occasions in 12 hours.

Addendum 1: Take an analog clock/timepiece where the hours are marked by dots or bars and no numerals are shown. Let the device be set to Indian Standard Time. Now hold the device upside down. Mentally advance the hour hand by 15º, i.e., the angle corresponding to a half hour. The device now shows Greenwich Mean Time/Universal Time. This is exact except that a.m./p.m. needs to be assigned. Check it out for a few positions and then try to prove that it always works.

Problem VII-2-M.3: **On Planetary Motion**

The situation is similar to the above except that the units are different. The angular velocities of planets A and B are $360/x$ degrees per day and $360/y$ degrees per day, respectively. Initially planet A races ahead and completes one full circuit in x days. By this time planet B has moved an angular distance of $\frac{360x}{y}$ degrees. Planet A now has to catch up with this at a relative angular velocity of $\frac{360}{x}$ degrees per day $-\frac{360}{y}$ degrees per day $= \frac{360(y-x)}{xy}$ degrees per day.

The time required for this is

$$\frac{360x}{y}\text{degrees} \div \frac{360(y-x)}{xy}\text{degrees per day} = \frac{x^2}{y-x}\text{days.}$$

We now need to add the time x to this to get the total time that has to elapse to have the two planets in a line with the sun again (being on the same side of the sun), which turns out to be

$$\left(x + \frac{x^2}{y-x}\right) \text{ days} = \frac{xy}{y-x} \text{ days.}$$

i) – (iv) The same holds true for the other cases mentioned.

Addendum 2: The problem about the planets can be related to our own solar system. The time interval between two occurrences of the same relative position is termed the Synodic period of either planet with respect to the other. Considering the Earth there can be two situations. Earth could be planet *B* as in the above problem. Then planet *A* would be a planet whose orbit lies inside that of the Earth (an 'inferior' planet). So if we know the orbital periods of Earth and the inferior planet we can calculate the synodic period. Alternatively and more practically, if we know Earth's orbital period and the synodic period of the other planet (this is observable from Earth), we can calculate the orbital period of the inferior planet. Denoting the synodic period by *S*, we have $S = \frac{xy}{y-x}$. This can be reformulated to give $x = \frac{Sy}{S+y}$.

If we take planet *A* to be Earth then planet *B* would be a planet moving in an orbit outside Earth's orbit (a 'superior' planet). As earlier, we could, from a knowledge of Earth's orbital period and the synodic period of the other planet (again observable from Earth), calculate the orbital period of the superior planet. The formula for *S* should now be reformulated to give $y = \frac{Sx}{(S-x)}$. It may be useful in this case to express the time periods in 'earth years' rather than 'earth days,' to simplify the computations.

The orbital and synodic periods of the planets in our neighbourhood can be obtained from the internet. The reader is invited to verify the statements made above. Despite the many simplifying assumptions made (circular and coplanar orbits and uniform orbital speed) there is good agreement.

APOLOGY FOR AN ERROR

We apologise for an error in the Middle School Problems of the March 2018 issue of At Right Angles. The solution that was presented for Problem VII-1-M.6 was actually the solution to Problem VII-1-M.7.

Also, there was an error in the statement of Problem VII-1-M.6. The corrected statement and the solution to this problem are given below.

Problem VII-1-M.6

Construct the locus of a point which moves so that it remains at equal distance from two given parallel straight lines l and m. Describe the locus in words.

Construct the locus of a point which moves so that it is a vertex of a trapezium of area 25 cm^2 one of whose parallel sides AB = 4 cm is at equal distance from two given parallel straight lines l and m which are at a distance of 5 cm from each other.

The first part of this question is extremely simple: the locus is a line parallel to the two given lines and situated midway between the two of them.

In the second part, side AB is situated on the above line. The parallel side is on either *l* or *m*. The base of the required trapezium is therefore 2.5 cm. Using the formula Area = 1/2 × base × sum of parallel sides, we find that the sum of the parallel sides is 20 cm. Since one of the parallel sides is 4 cm, the other side has length 16 cm. A point D can be chosen on either *l* or *m*, this fixes the remaining vertex C on this line at a distance of 16 cm on either side of D. So there are two possible trapeziums for each position of D on *l* and again on *m*.