Problem Corner

Alternative Solution to 'A Triangle Problem'

SANJIB RUDRA

Here is an alternative solution to the following problem which was studied in the November 2016 issue of *AtRiA*:

> *Two sides of a triangle have lengths* 6 *and* 10*, and the radius of the circumcircle of the triangle is* 12*. Find the length of the third side.*

Let the triangle be ABC, with sides $a = BC = 6$ and $b = CA = 10$ (Figure 1). The radius of the circumcircle is 12. We must find c, the length of side AB. Let $\measuredangle ACB = x$; then reflex $\measuredangle AOB = 2x$, so $\measuredangle AOB = 360^\circ - 2x$. From $\triangle ABC$ we get, using the cosine rule,

$$\cos x = \frac{6^2 + 10^2 - c^2}{2 \times 6 \times 10} = \frac{136 - c^2}{120}.$$

From $\triangle AOB$ we get, again using the cosine rule,

$$\cos(360^\circ - 2x) = \frac{12^2 + 12^2 - c^2}{2 \times 12 \times 12} = \frac{288 - c^2}{288}.$$

Hence:

$$\cos 2x = \frac{288 - c^2}{288}.$$

Since $\cos 2x = 2\cos^2 x - 1$, we have:

$$\frac{288 - c^2}{288} = 2\left(\frac{136 - c^2}{120}\right)^2 - 1,$$

$$\therefore \quad c^4 - 247c^2 + 4096 = 0, \qquad \text{(on simplification)}.$$

Keywords: *Triangle, circumradius, cosine rule*

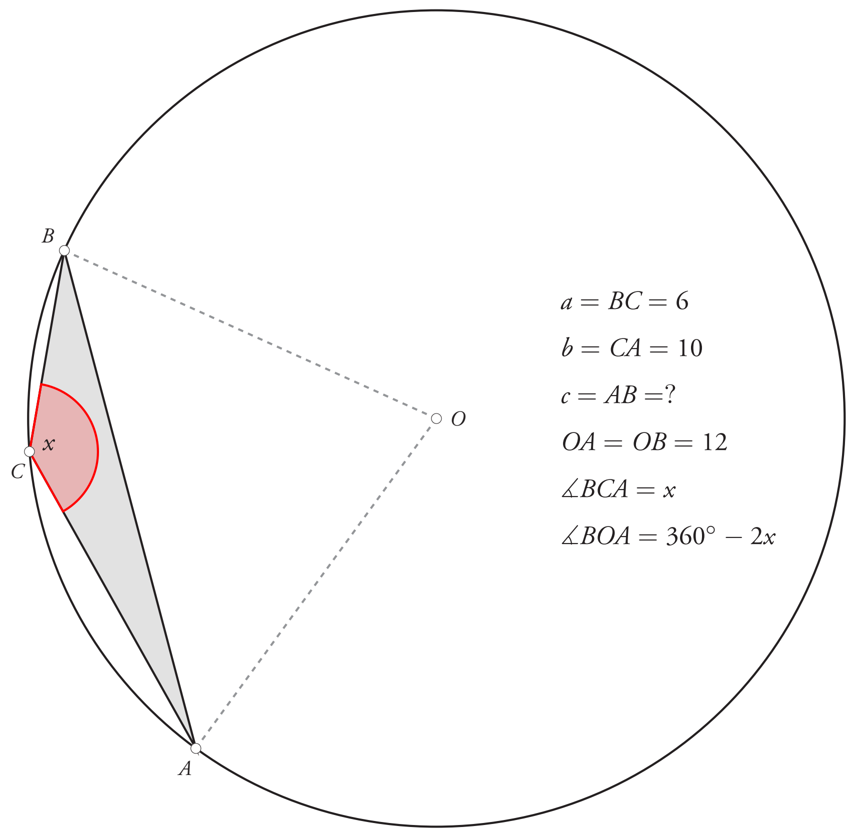

Figure 1.

Let $d = c^2$; then $d^2 - 247d + 4096 = 0$. The solution of this quadratic equation is

$$d = \frac{247 \pm 5\sqrt{1785}}{2},$$

giving $d \approx 229.123$ and $d \approx 17.877$. Hence, taking square roots,

$$c \approx 15.137, \qquad c \approx 4.228.$$

These are the two possible lengths of AB.