The Odd-Even Tale

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We are introduced to the concept of an even number and an odd number in primary school or even earlier. Any natural number divisible by 2 is even; if it is not, it is odd. The definition is extended to integers once we learn the arithmetic of negative whole numbers. Then we make simple observations such as: the sum of two even numbers is even, as is the sum of two odd numbers, and sum of an even number and an odd number is odd. Carrying on, we deduce that the sum of an even number of even numbers or odd numbers is even, and so is the sum of an odd number of even numbers, whereas the sum of an odd number of odd numbers is odd. This article aims at discussing some problems where these simple observations come into play.

Problem 1. At a party, each guest shakes hands with a certain number of guests. Is it true that the number of guests who have shaken hands with an odd number of guests is even? (It is taken for granted that each handshake is between precisely two persons; there are no handshakes featuring three or more hands!)

Solution to Problem 1. Let there be N guests present in the party. Suppose they are numbered $G_1, G_2, \ldots, G_N$. Let h_k, $1 \le k \le N$, be the number of handshakes performed by the guest G_k. Let

$$T = h_1 + h_2 + \cdots + h_N.$$

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First we show that T is even. Here is a nice way of proving it. Imagine that there is a counter placed in the party hall and initially it is set at zero. Whenever there is a handshake, the counter counts the number of hands involved in it. Thus at every handshake, the count on the counter increases by 2. Since initially the count is an even number (0 is even), the final count has to be an even number. Therefore, T is even. Without loss of generality, if we assume that $h_1, h_2, \ldots, h_M$, $M < N$, are odd then it follows that

$$h_1 + h_2 + \ldots + h_M = T - (h_{M+1} + \cdots + h_N)$$

is even, which shows that M must be even, for if M is odd, then on the left side we would have an odd number of odd numbers which necessarily add up to an odd number; on the other hand, all the quantities on the right side are even, consequently the right side is even. Hence M is even.

Problem 2. The numbers 1 through 10 are written in a row. Can the signs '$+$' and '$-$' be placed between them, so that the value of the resulting expression is 0?

Solution to Problem 2. We know that

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$$

If some of the '$+$' signs are replaced by '$-$' then the sum changes by an even number. (More specifically, the sum changes by twice the sum of the numbers thus altered, i.e., it changes by an even number.) Since the original sum is odd, no matter how many sign changes are made, the resulting sum will remain an odd number. Therefore it will not be possible to reach zero at any stage.

Problem 3. The numbers $1, 2, 3, \ldots, 2016, 2017$ are written on a blackboard. We decide to erase from the board any two numbers, and replace them with their positive difference. This process is continued till a single number remains on the blackboard. Can this number be zero?

Solution to Problem 3. Consider what happens at each stage to the sum of all the numbers on the blackboard before and after the number replacement. At any stage, let the numbers erased be a and b with $a > b$. Then the sum of all the numbers changes by

$$(a + b) - (a - b) = 2b,$$

an even number. *Hence, at each stage, the sum of the numbers changes by an even number*. Therefore the parity of the final sum and the initial sum will be the same. As the sum $1 + 2 + \cdots + 2016 + 2017$ is odd, the final number written on the board is odd too, and hence cannot be 0.

Problem 4. Can one form a 'magic square' with the first 36 prime numbers?

For the benefit of the reader, a "magic square" here means a 6×6 array of boxes, with a number in each box, and such that the sum of the numbers along any row, column, or diagonal is constant. The answer is NO. Why? Perhaps the reader would like to figure it out.

Problem 5. Let $a_1, a_2, \ldots, a_{2017}$ be a permutation of $1, 2, \ldots, 2017$ such that $a_k \neq k$ for every $k \in \{1, 2, \ldots, 2017\}$. Is the product

$$(a_1 - 1)(a_2 - 2) \cdots (a_{2017} - 2017)$$

even or odd?

Solution to Problem 5. There is an odd number of terms in the product. Suppose the product is odd. What can we say about each term? Each term must be odd. Thus for every k with $1 \leq k \leq 2017$, $a_k - k$

is odd. Now comes the crucial observation. It is motivated by the fact that $a_1, a_2, \ldots, a_{2017}$ is a permutation of $1, 2, \ldots, 2017$, and a permutation keeps the sum of the numbers unchanged. Therefore

$$a_1 + a_2 + \cdots + a_{2017} = 1 + 2 + \cdots + 2017,$$

and this can be re-written as

$$(a_1 - 1) + (a_2 - 2) + \cdots + (a_{2017} - 2017) = 0.$$

Note that the right side of this equality is even whereas each summand on the left side is odd, by assumption. But this cannot happen, because an odd number of odd numbers cannot add up to an even number. Thus we arrive at a contradiction and it arose from our assumption that the product is odd. Therefore the product must be even.

Note that the proposition is not true if the number of numbers we start with is even. For instance, if we had 2018 numbers, the claim could not have been made; in fact, it would have been false. To prove the falsity of the claim for an even number of numbers, we have to exhibit a permutation of this set of numbers for which the product defined in the statement of the problem is odd. We let the reader find such a permutation.

Problem 6. Let a, b and c be odd integers. Prove that the polynomial $ax^2 + bx + c$ does not have a rational root.

Solution to Problem 6. This is a very nice problem. We will solve it in two different ways. There is a reason for doing so, but we will not divulge it right now and rather let the suspense hang in the air. The first method is the textbook method: extract the roots and analyse them. The roots are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Assume that $b^2 - 4ac \geq 0$, so that the roots are real. If we want the roots to be rational, then $\sqrt{b^2 - 4ac}$ must be a positive integer. Let there exist a positive integer x such that

$$\sqrt{b^2 - 4ac} = x.$$

Observe that x is odd, since b^2 is odd and $4ac$ is even. Upon simplification, we get

$$ac = \left(\frac{b - x}{2}\right)\left(\frac{b + x}{2}\right).$$

Observe that both $b - x$ and $b + x$ are even numbers. Thus $\frac{b-x}{2}$ and $\frac{b+x}{2}$ are integers. In fact, they are both odd integers, because ac is odd. But then $b = \frac{b-x}{2} + \frac{b+x}{2}$ is even, contrary to the stated fact that it is odd. This contradiction shows that such a positive integer x does not exist.

Now we are ready for the second method. This is also a proof by contradiction, but it does not require extraction of roots. Here is how it runs. Suppose that the given equation has a rational root $x = \dfrac{p}{q}$, where p and q are integers, $q \neq 0$ and the greatest common divisor of p and q is 1, i.e., p and q are coprime. Thus

$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0.$$

Clearing the denominators leads to

$$ap^2 + bpq + cq^2 = 0.$$

Since p divides ap^2 as well as bpq, it must be that p divides cq^2. Similarly, q divides ap^2. But since p and q are coprime, we conclude that p divides c and q divides a. Thus p and q are odd and so is $ap^2 + bpq + cq^2$. This contradicts the above statement that $ap^2 + bpq + cq^2 = 0$, an even number.

Now the time has come to reveal the reason for discussing the second method. If we study the proof closely, we see that the vital part of the argument is to prove that p and q are odd. Once this is done, the rest of the proof relies on the fact that the sum of an odd number of odd numbers cannot be even. Nowhere did we use the fact that we are dealing with a quadratic polynomial. All that mattered in the end was that an odd number of terms were present in the expression. This opens up the possibility of generalising the proposition to polynomials of arbitrary degree with odd coefficients and having an odd number of terms. To put it in precise mathematical terms, pick an even number k and consider a finite sequence of natural numbers $n_0 < n_1 < \cdots < n_k$ and odd integers $a_{n_0}, a_{n_1}, \cdots, a_{n_k}$. Construct the polynomial

$$a_{n_k}x^{n_k} + a_{n_{k-1}}x^{n_{k-1}} + \cdots + a_{n_1}x^{n_1} + a_{n_0}.$$

This polynomial has $k+1$ terms, which is odd because k is even. This polynomial does not have a rational root. Why? Let us emulate the argument that we used for the quadratic. It is evident that zero is not a root of the polynomial. If possible, let there be a non-zero rational root $x = \frac{p}{q}$, where p and q are coprime integers and $q \neq 0$. Then

$$a_{n_k}\left(\frac{p}{q}\right)^{n_k} + a_{n_{k-1}}\left(\frac{p}{q}\right)^{n_{k-1}} + \cdots + a_{n_1}\left(\frac{p}{q}\right)^{n_1} + a_{n_0} = 0.$$

Multiplying both sides of the equation by q^{n_k} leads to

$$a_{n_k}p^{n_k} + a_{n_{k-1}}p^{n_{k-1}}q^{n_k - n_{k-1}} + \cdots + a_{n_1}p^{n_1}q^{n_k - n_1} + a_{n_0}q^{n_k} = 0.$$

As before, we observe that p divides a_{n_0} and q divides a_{n_k}, hence both are odd. Thus every term on the left hand side of the preceding equation is odd and there are an odd number of them. Therefore the sum of these terms cannot be zero. This contradiction shows that the polynomial cannot have a rational root.

Note that we did not explicitly extract the roots of the polynomial equation in order to carry out the analysis and complete the argument. The reader may be aware that finding roots of a general polynomial is a herculean task. The second method not only overcomes this difficulty, it also reduces the complexity of the problem to a great extent.

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