Problem Corner

Adventures in Problem Solving
A Problem About Divisors

SHAILESH SHIRALI

In this edition of 'Adventures' we study a curious problem concerning the divisors of a certain number. Partial information has been provided about the divisors and on that basis we are required to find the number. The information provided seems at first sight to be meagre in the extreme. But strangely, it suffices to make progress. Read on!

The problem

For an arbitrary positive integer n, list its divisors in increasing order, starting with 1 and ending with n. Let the divisors be $d_1, d_2, d_3, \ldots, n$ where $d_1 = 1$ and $d_1 < d_2 < d_3 < \cdots < n$. Find all possible values of n for which the following property is satisfied:

$$d_8 + d_{10} + d_{11} = n. \tag{1}$$

In other words, find all possible values of n for which the 8-th, 10-th and 11-th divisors of n add up to n itself.

The problem looks formidable (which Olympiad problem does not?), but, as we shall see, there are enough clues to solve it. In fact, we shall uncover a surprising conclusion.

Two observations

At the start, we make a simple yet easily missed observation which holds the key to the solution of this problem:

I: If d is a divisor of a positive integer n, then n/d too is a divisor of n.

Keywords: *Divisor, constraints, combinations, systematic reasoning*

The divisors d and n/d are called *complementary divisors* of n. For example, 2 and 5 are a pair of complementary divisors of 10.

Our second observation is a well-known theorem of elementary number theory:

II: Let the prime factorization of a positive integer n be $n = p^u \cdot q^v \cdot r^w \cdots$ where $p, q, r, \ldots$ are distinct prime numbers, and $u, v, w, \ldots$ are positive integers. Then the number of divisors of n is given by the product

$$(u+1) \cdot (v+1) \cdot (w+1) \cdots. \tag{2}$$

Example 1: Consider the integer 12; its factorization into primes is $12 = 2^2 \cdot 3$. We expect the number of divisors of 12 to be $(2+1) \cdot (1+1) = 6$. And 12 does indeed have 6 divisors (they are: $1, 2, 3, 4, 6, 12$).

Example 2: Consider the integer 30; its factorization into primes is $30 = 2 \cdot 3 \cdot 5$. We expect the number of divisors of 30 to be $(1+1) \cdot (1+1) \cdot (1+1) = 8$. And 30 does indeed have 8 divisors (they are: $1, 2, 3, 5, 6, 10, 15, 30$).

It is an easy exercise to prove the formula. We only need to use these facts: (i) a divisor of n must be made up of the same primes that divide n; (ii) the power to which a prime number divides the divisor cannot exceed the power to which that prime number divides n itself.

Solution to the problem

Let $a = n/d_8$, $b = n/d_{10}$ and $c = n/d_{11}$. Then $1 < c < b < a$, and the condition $d_8 + d_{10} + d_{11} = n$ translates to:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1. \tag{3}$$

The first task, clearly, is to identify all positive integer solutions of the above equation. We shall show that with the condition $c < b < a$, there is just one solution.

Suppose that $c \geq 3$. Then $a > b > c \geq 3$, hence:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{3} + \frac{1}{3} + \frac{1}{3},$$

$$\therefore \ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1. \tag{4}$$

So if $c \geq 3$, then relation (3) cannot be satisfied. Since $c > 1$, it follows that $c = 2$, i.e., $d_{11} = n/2$. *Hence, 2 is one of the prime divisors of n.*

Since $c = 2$, we get:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2}, \tag{5}$$

and $a > b > 2$, i.e., $a > b \geq 3$.

Suppose that $b \geq 4$. Then $a > b \geq 4$, hence:

$$\frac{1}{a} + \frac{1}{b} < \frac{1}{4} + \frac{1}{4},$$

$$\therefore \ \frac{1}{a} + \frac{1}{b} < \frac{1}{2}. \tag{6}$$

So if $b \geq 4$, then relation (5) cannot be satisfied. Since $b > 2$ (obtained above), it follows that $b = 3$, i.e., $d_{10} = n/3$. *Hence, 3 is one of the prime divisors of n.*

Having obtained $c = 2$ and $b = 3$, we get a by substitution:

$$\frac{1}{a} = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, \qquad \therefore\ a = 6.$$

Let us summarize what we have obtained till now: 2, 3 *and* 6 *are divisors of n, and*:

$$d_1 = 1, \quad d_2 = 2, \quad d_3 = 3, \quad \ldots, \quad d_8 = \frac{n}{6}, \quad d_{10} = \frac{n}{3}, \quad d_{11} = \frac{n}{2}. \tag{7}$$

Since 2 and 3 are prime divisors of n, it follows that n has the following form:

$$n = 2^u \cdot 3^v \cdot w, \tag{8}$$

where u, v are positive integers, and w is a positive integer not divisible by either 2 or 3, i.e., $\gcd(w, 6) = 1$.

Between $n/2$ and n, there can clearly be no further divisors of n; do you see why? Hence the next divisor after d_{11} must be n itself; that is, $d_{12} = n$. So we obtain another important property:

$$n \textit{ has precisely } 12 \textit{ divisors.} \tag{9}$$

Next we ask: Which integers have exactly 12 divisors? On listing the different ways in which 12 can be written as a product of integers exceeding 1,

$$12 = 6 \times 2 = 4 \times 3 = 3 \times 2 \times 2, \tag{10}$$

we deduce that the positive integers which have exactly 12 divisors are of the following four kinds:

$$p^{11}, \quad p^5 \cdot q, \quad p^3 \cdot q^2, \quad p^2 \cdot q \cdot r, \tag{11}$$

where p, q, r are unequal primes. For example, each of the following has exactly 12 divisors:

$$2^{11}, \quad 3^{11}, \quad 2^5 \times 3, \quad 3^5 \times 2, \quad 2^3 \times 3^2, \quad 3^3 \times 2^2, \quad 2^2 \times 3 \times 5, \quad \ldots.$$

Moreover, the forms listed in (11) cover all the possibilities.

We see that there are infinitely many integers having 12 divisors. From this infinite collection, we need to select those which satisfy the conditions in our problem.

There has been no mention of d_9 till now. But we can say this: Since $d_8 = n/6$ and $d_{10} = n/3$, it must be that $d_9 = n/4$ **or** $d_9 = n/5$. Hence n has either 4 or 5 as a divisor — but not both.

Combining this deduction with what we got earlier, we see that precisely one of the two possibilities listed below holds:

(a) $n = 2^u \cdot 3^v \cdot r$ where u, v are positive integers with $u \geq 2$, and r is a positive integer not divisible by 2, 3 or 5; **OR**:

(b) $n = 2 \cdot 3^v \cdot 5^w \cdot s$ where v, w are positive integers and s is a positive integer not divisible by 2, 3 or 5.

To this finding, we bring the fact that n must have one of the forms listed in (11). It follows that n must be one of the following:

- $n = 2^5 \times 3 = 96$;
- $n = 2^3 \times 3^2 = 72$;
- $n = 3^3 \times 2^2 = 108$;
- $n = 2^2 \times 3 \times w = 12w$ where w is a prime number greater than 5;

- $n = 3^2 \times 2 \times 5 = 90$;
- $n = 5^2 \times 2 \times 3 = 150$.

Hence the integers which satisfy the stated condition are precisely the following:

$$72, \quad 90, \quad 96, \quad 108, \quad 150, \quad 12w,$$

where w is any prime number greater than 5.

Here are three sample verifications. Observe that the relation $d_8 + d_{10} = d_{11}$ holds in each case, and this is true by virtue of the equality $1/6 + 1/3 = 1/2$; or, equivalently, $n/6 + n/3 = n/2$.

- $n = 90$: The divisors of n are
$$1,\ 2,\ 3,\ 5,\ 6,\ 9,\ 10,\ 15,\ 18,\ 30,\ 45,\ 90,$$
and we may check that $15 + 30 + 45 = 90$.
- $n = 108$: The divisors of n are
$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12, 18,\ 27,\ 36,\ 54,\ 108,$$
and we may check that $18 + 36 + 54 = 108$.
- $n = 12 \times 11 = 132$ (the form $12w$ with $w = 11$): The divisors of n are
$$1,\ 2,\ 3,\ 4,\ 6,\ 11,\ 12,\ 22,\ 33,\ 44,\ 66,\ 132,$$
and we may check that $22 + 44 + 66 = 132$.

We have discovered a surprising fact: There are infinitely many integers which satisfy the stated condition!

A problem for you to tackle …

Now we close with a problem for you; it is a small variation of the problem we have solved, but the outcome turns out to be quite different.

> For an arbitrary positive integer n, list its divisors in increasing order, starting with 1 and ending with n. Let the divisors be $d_1, d_2, d_3, \ldots, n$ where $d_1 = 1$ and $d_1 < d_2 < d_3 < \cdots < n$. Find all possible values of n for which the following property is satisfied: $d_7 + d_{10} + d_{11} = n$.

SHAILESH SHIRALI is the Director of Sahyadri School (KFI), Pune, and heads the Community Mathematics Centre based in Rishi Valley School (AP) and Sahyadri School KFI. He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.