Problem Corner

The Three Circles Problem

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

In this article, we study the following problem. *Three circles of equal radius r are centred at the vertices of an equilateral triangle ABC with side $2a$. Here we assume that $r > a$. Find the area of the three-sided region DEF enclosed by all three circles, in terms of r and a.* (See Figure 1.)

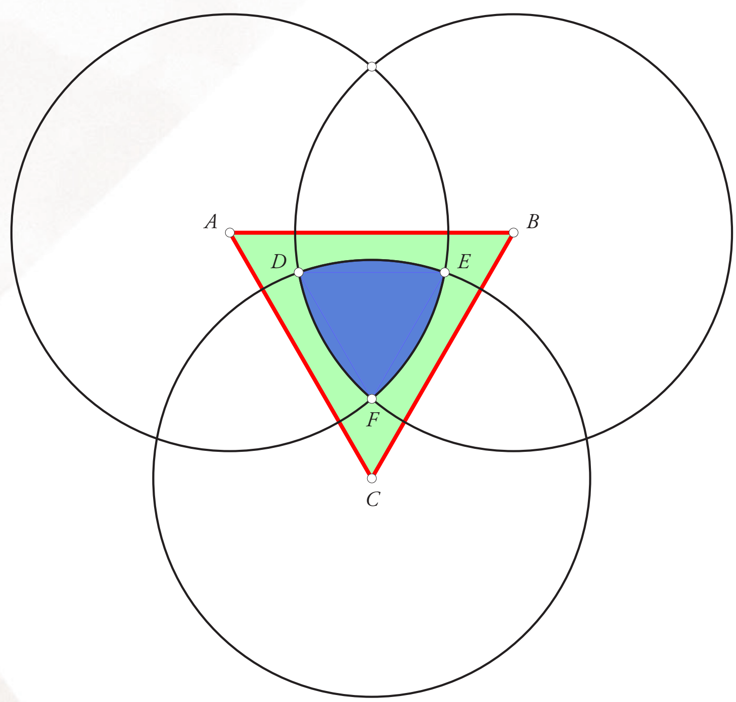

Figure 1.

Solution. We carry out the analysis as shown below.

Keywords: *Circles, intersection, area*

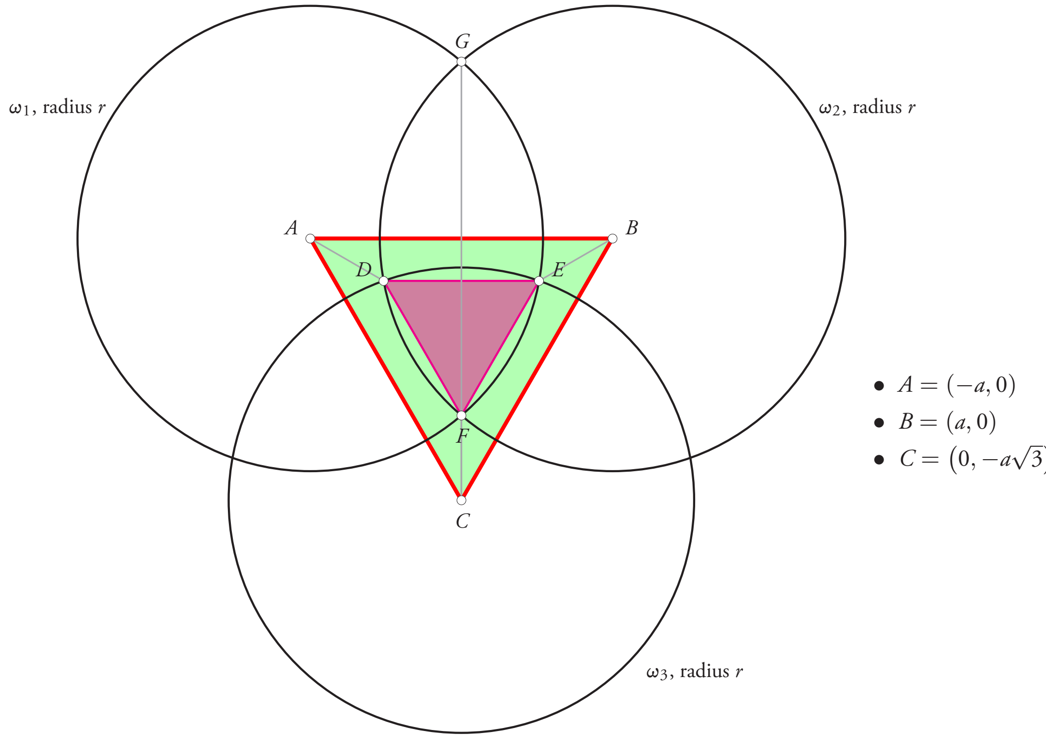

Figure 2.

(1) Mark points D, E, F, G as shown. Using Pythagoras's theorem, we obtain $FG = 2\sqrt{r^2 - a^2}$. Let $2d$ be the length of DE. We must first find d in terms of r and a.

(2) Assign coordinates as shown:

$$A = (-a, 0), \quad B = (a, 0), \quad C = \left(0, -a\sqrt{3}\right).$$

The equations of the three circles then are:

$$\begin{aligned} \omega_1: \quad & (x+a)^2 + y^2 &= r^2, \\ \omega_2: \quad & (x-a)^2 + y^2 &= r^2, \\ \omega_3: \quad & x^2 + \left(y + a\sqrt{3}\right)^2 &= r^2. \end{aligned}$$

(3) The coordinates of points D, E, F, G can now be worked out by solving pairs of simultaneous equations. Here is what we get:

$$D = \left(\frac{a - \sqrt{3\left(r^2 - a^2\right)}}{2}, \frac{-a\sqrt{3} + \sqrt{r^2 - a^2}}{2} \right),$$

$$E = \left(\frac{-a + \sqrt{3\left(r^2 - a^2\right)}}{2}, \frac{-a\sqrt{3} + \sqrt{r^2 - a^2}}{2} \right),$$

$$F = \left(0, \ -\sqrt{r^2 - a^2}\right),$$

$$G = \left(0, \ \sqrt{r^2 - a^2}\right).$$

(4) The length of DE can now be worked out from the coordinates of D and E:

$$DE = \sqrt{3\left(r^2 - a^2\right)} - a.$$

(5) The area of triangle DEF can now be worked out using the above expression:

$$\begin{aligned}\text{Area of } \triangle DEF &= \frac{\sqrt{3}}{4}\left(\sqrt{3\left(r^2 - a^2\right)} - a\right)^2 \\ &= \frac{3r^2\sqrt{3} - 2a^2\sqrt{3} - 6a\sqrt{r^2 - a^2}}{4}.\end{aligned}$$

(6) Next, we find $\theta = \measuredangle DCE$, using the length of DE:

$$\begin{aligned}\sin\theta &= \frac{DE/2}{r} \\ &= \frac{\sqrt{3\left(r^2 - a^2\right)} - a}{2r}.\end{aligned}$$

(7) This allows us to find the area of the minor segment bounded by segment DE and circle ω_3:

$$\text{Area of segment } D\omega_3 E = \frac{r^2(\theta - \sin\theta)}{2}.$$

(8) Finally, the area of the region DEF is given by:

$$\text{Area of region } DEF = \text{Area of } \triangle DEF + 3 \cdot \text{Area of segment } D\omega_3 E.$$

This simplifies, after a lot of work, to:

$$\begin{aligned}&-\frac{3}{2}a\sqrt{r^2 - a^2} + \frac{3}{4}a\sqrt{2a\left(\sqrt{3\left(r^2 - a^2\right)} + a\right) + r^2} \\ &- \frac{3}{4}\sqrt{3}\sqrt{\left(r^2 - a^2\right)\left(2a\left(\sqrt{3\left(r^2 - a^2\right)} + a\right) + r^2\right)} \\ &+ 3r^2\sin^{-1}\left(\frac{\sqrt{3\left(r^2 - a^2\right)} - a}{2r}\right) - \frac{1}{2}\sqrt{3}a^2 + \frac{3\sqrt{3}r^2}{4}\end{aligned}$$

This is the required area.

(9) For $r = 10, a = 6$, we get:

$$\begin{aligned}\text{Area of region } DEF &= 36\left(\sqrt{3} - 4\right) + 300\sin^{-1}\left(\frac{1}{10}\left(4\sqrt{3} - 3\right)\right) \\ &\approx 39.4628 \text{ square units.}\end{aligned}$$