Problem About a Finite Sequence

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The following problem appeared in the International Mathematical Olympiad (IMO) of 1977, held that year in (the former Republic of) Yugoslavia (see [1]; it is the second problem in the paper):

> *In a finite sequence of real numbers, the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.*

It is not difficult to show that such a sequence cannot have more than 16 terms. Suppose there is a 17-term sequence $a_1, a_2, a_3, \ldots, a_{16}, a_{17}$ having the stated property. We shall show that this leads to a contradiction. We use the terms of the sequence to construct the following 7×11 matrix:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}

As per the given information, the sum of the terms in each row is positive, hence the sum of all the terms in the entire matrix is positive. Also as per the given information, the sum of the terms

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in each column is negative, hence the sum of all the terms in the entire matrix is negative. We have an obvious contradiction. Hence there cannot exist a 17-term sequence having the stated property.

The interesting question now arises: *Does there exist a 16-term sequence having such a property, and if so, how do we construct such a sequence?* Here is one line of reasoning which helps us find such a sequence. Suppose that $a_1, a_2, a_3, \ldots, a_{16}$ is a 16-term sequence with the stated property. Let us now see what can be gleaned from this information. Let b_k be the sum of the first k terms of the a-sequence, i.e.,

$$b_k = a_1 + a_2 + a_3 + \cdots + a_k.$$

Observe that there are 16 numbers $b_1, b_2, b_3, \ldots, b_{16}$. Can we state which of these is the smallest and which one is the largest? It turns out that the complete chain of order relations can be deduced using only the stated property. For we have:

$$0 < b_{11} < b_4 < b_{15} < b_8 < b_1 < b_{12} < b_5 < b_{16} < b_9 < b_2 < b_{13} < b_6.$$

For example, $b_{11} < b_4$ is true because $a_5 + a_6 + \cdots + a_{10} + a_{11} < 0$; similarly for the other relations. We also have:

$$b_{10} < b_3 < b_{14} < b_7.$$

Finally we have $b_7 < 0$. This means that

$$b_{10} < b_3 < b_{14} < b_7 < 0 < b_{11} < b_4 < b_{15} < b_8 < b_1 < b_{12} < b_5 < b_{16} < b_9 < b_2 < b_{13} < b_6.$$

So we have deduced the full chain of order relations! Quite remarkable. . .

These relations now suggest a way of constructing a sequence with the required property: *assign to the quantities* $b_{10}, b_3, b_{14}, b_7, b_{11}, b_4, b_{15}, b_8, \ldots, b_{13}, b_6$ ***any*** 16 *numbers in increasing order, the first four of them being negative, and then solve for the unknown quantities* $a_1, a_2, a_3, \ldots, a_{16}$. For example, we may put $b_{10} = -4$, $b_3 = -3$, $b_{14} = -2$, $b_7 = -1$, $b_{11} = 1$, $b_4 = 2$, $b_{15} = 3$, $b_8 = 4, \ldots, b_6 = 12$. Here are the values displayed in tabular form:

b_{10}	b_3	b_{14}	b_7	b_{11}	b_4	b_{15}	b_8	b_1	b_{12}	b_5	b_{16}	b_9	b_2	b_{13}	b_6
– 4	– 3	– 2	– 1	1	2	3	4	5	6	7	8	9	10	11	12

Or, rearranging the entries and displaying them in a more convenient form:

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
5	10	– 3	2	7	12	– 1	4	9	– 4	1	6	11	– 2	3	8

These yield the following, in turn:

$$a_1 = b_1 = 5;$$

$$a_2 = b_2 - b_1 = 10 - 5 = 5;$$

$$a_3 = b_3 - b_2 = -3 - 10 = -13;$$

$$a_4 = b_4 - b_3 = 2 + 3 = 5;$$

$$a_5 = b_5 - b_4 = 7 - 2 = 5;$$

$$a_6 = b_6 - b_5 = 12 - 7 = 5;$$

$$a_7 = b_7 - b_6 = -1-12 = -13;$$

$$a_8 = b_8 - b_7 = 4 - (-1) = 5;$$

and so on. Proceeding systematically in this manner, we obtain all the numbers. Here are the resulting values:

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
5	5	– 13	5	5	5	– 13	5	5	– 13	5	5	5	– 13	5	5

Isn't it curious that the sequence assumes only two different values? (Of course, the values are decided by the choice of the 16 numbers made earlier.)

It should be clear from the way we have found the values of the a_i that, regardless of what values we give to the quantities $b_{10}, b_3, b_{14}, b_7, b_{11}, b_4, b_{15}, b_8, \ldots, b_{13}, b_6$, we will always obtain a-values that fit the various equations.

References

[1] International Mathematical Olympiad, https://www.imo-official.org/year_info.aspx?year=1977

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