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Creating Trigonometric Tables

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Have you tried to create your own trigonometric tables and thus join the likes of several mathematician astronomers of the past who created tables of trigonometric functions for their astronomical calculations?

Sine tables in Bhāskarācārya's Siddhānta-Śiromaṇi

The great 12th century Indian mathematician and astronomer, Bhāskarācārya II, describes the creation of sine tables in his magnum opus, the Siddhānta-Śiromaṇi, in a section named Jyotpatti [1]. The very name Jyotpatti means 'jyānām utpattiḥ', i.e., creation or generation of sine tables. Jyotpatti is a part of Golādhyāya (dealing with Trigonometry), which is one of the four major parts of the Siddhānta-Śiromaṇi, the other three being Līlāvatī (dealing with Arithmetic), Bījagaṇita (dealing with Algebra) and Grahagaṇita (dealing with Planetary Motion).

Bhāskarācārya describes methods for creating several sine tables of varying granularities (i.e., different angle intervals: 3°, 3.75° and so on) [2], [3]. Among these is a table of approximate sine values for each integral angle in the quadrant and a table of exact sine values for all angles that are multiples of 3° in the quadrant. Bhāskara states that sine values can be determined in several ways and lists various identities to illustrate the point. Among these, he specifically highlights the usefulness of the following identities in the creation of these tables

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$ (1) (verse 21)
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$ (2) (verse 22)

Keywords: trigonometry, table, sine table, astronomy, Bhāskarācārya II, Siddhānta-Śiromaṇi

Creating Sine tables for angles at 3° intervals

Let us see how we can create a sine table for angles that are multiples of 3° using just identity (2) and the values of sin 90°, sin 45° and sin 30°.

Now, suppose the exact value of sin 3° were known. Substituting A=90 and B=3 in identity (2), the value of sin 87° can be determined (since we already know that sin 90° = 1). Now, substituting A=87 and B=3 will yield the value of sin 84°. Proceeding in this manner, the sine values for the other angles can be successively found.

The next obvious question is: how do we find the value of sin 3°?

Using A=45 and B=30 in identity (2), we can find the value of sin 15°.

If we could find the value of sin 18°, then substituting A=18 and B=15 in identity (2) would give us the value of sin 3°.

The rest of this article deals with a very elegant geometrical method to find the exact value of sin 18° [4].

The Sine of 18°

The value of sin18° is given by Bhāskarācārya in verse 9 of the Jyotpatti. The verse is:

त्रिज्याकृतीषुघातान्मूलं त्रिज्योनितं चतुर्भक्तम् ।
अष्टादशभागानां जीवा स्पष्टा भवत्येवम् ॥ ९ ॥

(Please see the appendix for the translation of this verse and another verse which appears later in the article.)

The verse tells us that $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$. Let us now see how to arrive at this result.

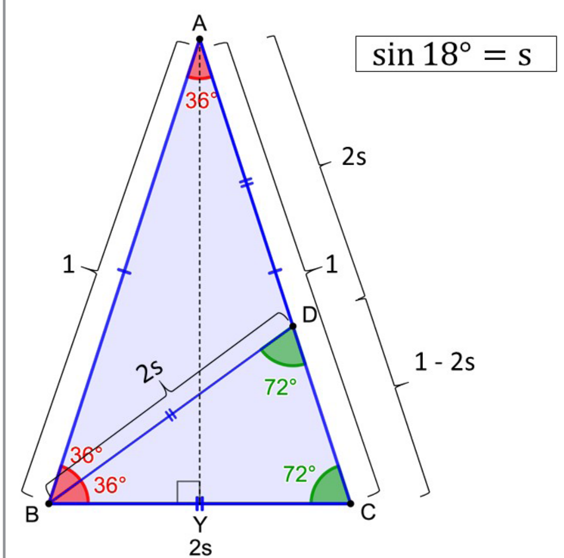

Figure 1.

Consider an isosceles triangle ΔABC (Figure 1) in which ∠BAC = 36° and AB = AC = 1 unit. Using the Angle Sum Property, we get ∠ABC = ∠ACB = 72°.

Construct altitude AY from vertex A to side BC. It is easy to see that BY = YC = s, where s = sin 18°, and thus BC = 2s.

Construct BD, the angular bisector of ∠ABC. This gives us two triangles, ΔABD and ΔBDC, both of which are isosceles. BD and AD are the equal sides of ΔABD and BD and BC are the equal sides of ΔBDC. Therefore, AD = BD = BC = 2s and DC = AC - AD = 1 - 2s.

Further, we have the triangle similarity ΔBDC ~ ΔABC. Using the similarity ratio, we get,

$$\frac{DC}{BC} = \frac{BC}{AC}$$
$$\Rightarrow \frac{1-2s}{2s} = \frac{2s}{1},$$
$$\Rightarrow 4s^2 = 1-2s,$$
$$\Rightarrow 4s^2 + 2s - 1 = 0.$$

Thus, we have a quadratic equation whose root is s.

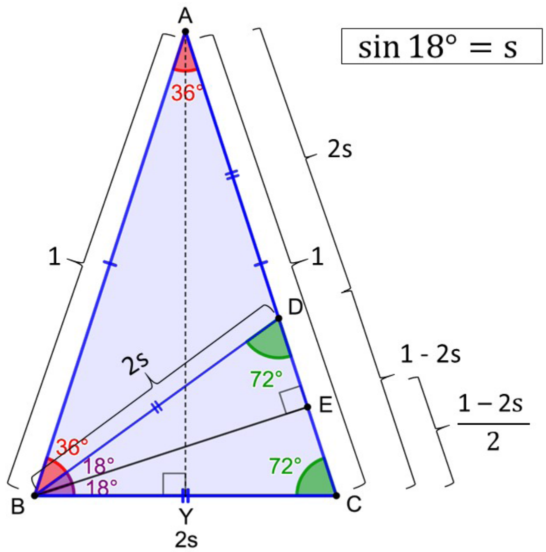

Figure 2.

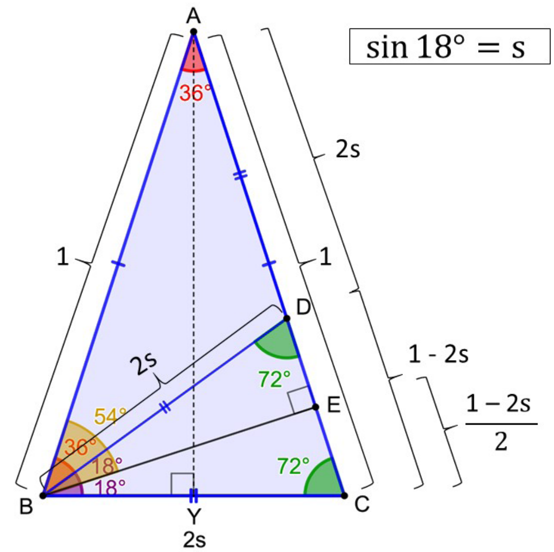

Figure 3.

Applying the quadratic formula, we get s, which is the value of sin 18°, to be $\frac{\sqrt{5}-1}{4}$.

Using a slight modification to the previous method, we can arrive at the quadratic equation in a different way.

Construct altitude BE of ΔBDC (Figure 2). Consider the right triangle ΔBEC. In this triangle,

$$\sin\ 18^\circ = s = \frac{EC}{BC} = \frac{1-2s}{2} \times \frac{1}{2s}.$$

Rearranging the terms, we once again arrive at the quadratic equation $4s^2 + 2s - 1 = 0$.

This second method yields us something more; we can find the exact value of sin 54° too! Further, we get a nice relation between sin 18° and sin 54°.

In the right triangle ΔAEB, (Figure 3)

$$\sin 54^\circ = \frac{AE}{AB} = 2s + \frac{1-2s}{2} = \frac{1}{2} + s = \frac{1}{2} + \sin 18^\circ = \frac{\sqrt{5}+1}{4}.$$

Conclusion

It is quite exciting to note that we can create our own trigonometric tables using simple high school mathematics!

It is also fascinating to see how these results were captured so compactly and beautifully in verse form by the great Indian mathematicians of the past.

Drawing inspiration from the verses in the Jyotpatti section, we conclude with a humble attempt at a verse* capturing the value of sin 54°as well as its relation with sin18°.

त्रिज्याकृतीषुघातात्
मूलं त्रिज्याधिकं चतुर्भक्तम् ।
त्रिज्यार्धं वसुविधुलव-
गुणसहितं युगशरांशज्या॥

(*) Thanks to Prof. K. Ramasubramanian and Dr. K. Mahesh (both of IIT Bombay) for reviewing and correcting this verse

The first two lines of this verse capture the fact that sin 54°= $\frac{\sqrt{5}+1}{4}$ and the last two lines capture the relation that $\sin 54^\circ = \frac{1}{2} + \sin 18^\circ$.

Appendix: Translations of the Sanskrit Verses

त्रिज्याकृतीषुघातान्मूलं त्रिज्योनितं चतुर्भक्तम् ।
अष्टादशभागानां जीवा स्पष्टा भवत्येवम् ॥ ९ ॥

त्रिज्या - Radius of the circle (R)

कृति – square (here, square of the radius)

इषु – 5 (in the Bhūtasaṃkhyā system [5])

घात – multiplied by

त्रिज्याकृतीषुघात – five times the square of the radius

मूलम् – square root (here, square root of five times the square of the radius)

त्रिज्या-ऊनितम् – reduced by R

चतुर्भक्तम् – divided by 4

अष्टादश - 18

जीवा – Indian Sine = Rsine

अष्टादशभागानां जीवा – Rsine 18°

Translation: The square root of five times the square of the radius, reduced by the radius, divided by four, is the exact value of the Rsine of 18°.

$$\text{i.e., Rsine } 18^\circ = \frac{\sqrt{5R^2} - R}{4}$$

$$\Rightarrow \sin 18^\circ = \frac{\sqrt{5} - 1}{4}$$

त्रिज्याकृतीषुघातान्मूलं त्रिज्याधिकं चतुर्भक्तम् ।
त्रिज्यार्धं वसुविधुलवगुणसहितं युगशरांशज्या॥

त्रिज्या – Radius of the circle (R)

कृति – square (here, square of the radius)

इषु – 5 (in the Bhūtasaṃkhyā system [5])

घात – multiplied by

त्रिज्याकृतीषुघात – five times the square of the radius

मूलम् – square root (here, square root of five times the square of the radius)

त्रिज्या-अधिकम् – increased by R

चतुर्भक्तम् – divided by 4

त्रिज्या-अर्धम् – half of R

वसुविधु – 18 (in the Bhūtasaṃkhyā system)

गुण – Indian Sine (Rsine)

वसुविधुलवगुण – Rsine 18°

सहितम् – along with (added to)

त्रिज्यार्धं वसुविधुलवगुणसहितं – half of R added to Rsine 18°

युगशर – 54 (in the Bhūtasaṃkhyā system)

ज्या - Indian Sine (Rsine)

युगशरांशज्या – Rsine 54°

Translation: The square root of five times the square of the radius, increased by the radius, divided by four, is the value of the Rsine of 54°; and this is equal to the Rsine of 18° added to half the radius.

$$\text{i.e., Rsine } 54^\circ = \frac{\sqrt{5R^2} + R}{4} = \frac{R}{2} + \textbf{Rsine } \mathbf{18^\circ}$$

$$\Rightarrow \sin 54^\circ = \frac{\sqrt{5} + 1}{4} = \frac{1}{2} + \mathbf{\sin 18^\circ}$$

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