Functional Equations

Part 1

A *functional equation* is an equation where the unknown is a function rather than a variable. It may happen that one knows only a certain property of a function; e.g. that it is even, i.e., $f(x) = f(-x)$ for all x; or that it is additive, i.e., $f(x + y) = f(x) + f(y)$ for all x, y; and so on. The question that then arises is, what functions exist with the stated property? Is there just one such function? In this two-part article, we answer such questions for certain types of functional equations. Functional equations are used to model behaviour in engineering fields (e.g., Shannon's entropy in Information Theory) and the social sciences. They are also of use in the study of difference equations.

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In Part I of this two-part article, student Shuborno Das goes into the topic of Functional Equations in which the unknown quantities are functions rather than numbers. He draws deeply on concepts related to the topic of Functions which is covered in Standards 11-12. Each new concept that is introduced is clearly defined, with plenty of examples and explanations. A delightful way to learn more about functions and operate flexibly with them.

What are functional equations?

Functions. A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output [1]. In other words, a function is a mapping between two sets A and B where each element in A maps to a unique element in B. This tells us that $f(a)$ cannot be b and c at the same time, where $b \neq c$. A function can be considered as an input-output machine, which takes an input and gives an output (Figure 1). The set of values which the machine can take as input is the *domain*, and the set of values which the machine gives as output is the *range*. Mathematically, a function f which maps elements of set A into set B is written as $f : A \to B$.

Example.

- $f(x) = x^2$ for all real x. Here, the domain is the set of real numbers and the range is the set of all nonnegative real numbers.
- $f(x) = \sin(x)$ for all real x. Here, the domain is the set of real numbers and the range is $[-1, 1]$.

Co-domain of a function is the possible values which a function can give as output. Range is a subset of co-domain.

Keywords: Functional equation, function, domain, range, injective function, surjective function

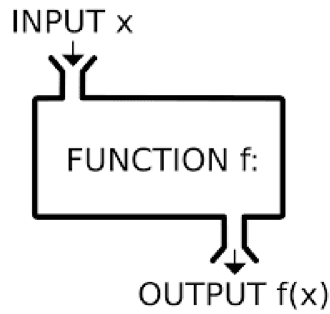


Figure 1. Function as input-output machine [1]

Functional Equations. A functional equation is an equation in which the unknown is a function of one or more variables. Often, the functional equation relates the value of a function (or functions) at some point with its values at other points [1]. Also, since it involves functions, the domain and range must also be specified unlike the case of more common equations.

Example.

- Suppose $f: A \to B$ such that $f(x+1) = f(x) + 1$ for all x in A.
- Suppose $f: \mathbb{R} \to \mathbb{R}$ such that $f(2f(x) + f(y)) = 2x + f(y)$ for all $x, y \in \mathbb{R}$.

Some properties of functions
We will discuss some properties of functions which are useful in solving functional equations.

Injective. An *injective function* or *injection* or *one-to-one function* is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its co-domain. In other words, every element of the function's co-domain is the image of one element of its domain [1].

Definition. If a function $f: A \to B$ has the property that the equality $f(a) = f(b)$ implies $a = b$, then f is said to be injective.

Let's get to some basic examples of recognising the injective property of a function.

- Problem 1: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = \sin x$. Is f injective?

Let $f(a) = f(b)$.

$$\sin a = f(a) = f(b) = \sin b.$$

Consider $a = 30°$ and $b = 150°$, $\sin a = \sin b$ but $a \neq b$. Hence f is not injective.

- Problem 2: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = x^3$. Is f injective?

Let $f(a) = f(b)$.

$$a^3 = f(a) = f(b) = b^3.$$

Since a, b are real, it follows that $a = b$. Hence f is injective.

Some facts regarding injectivity:

- Noticing a term of x not inside f(some variable) is pretty helpful. For example, in Problem 2, RHS had a term of x not involving any f, which helped us show the injectivity of f.

- It is important to check the range and domain while proving whether a function is injective or not. If the domain and range of f was complex in the previous example, then f wouldn't be injective.
- Using injectivity, we can reduce composite functions. For example, if we have $f(f(x^2)) = f(f(x)^2)$ and if f is injective then we can get $f(x^2) = f(x)^2$. That is, function value of x^2 is equal to the square of the function value of x.

Surjective. A *surjective function* or *onto function* is a function in which the elements of the domain map to the entire set of elements in the range [1].

Definition. If a function $f: A \to B$ has the property that for every b in B, there exists a value of a in A such that $f(a) = b$, then f is said to be surjective.

This tells us that for any element in the co-domain, we will have at least one element in the domain mapping to it. Let's get to some basic examples of recognising the surjective property of a function.

- Problem 3: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = \sin x$. Is f surjective?

We have

$$-1 \le \sin x \le 1,$$

therefore f is not surjective.

- Problem 4: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = x^3$. Is f surjective?

  Suppose $a^3 = f(a) = b$, then $a = \sqrt[3]{b}$ (we are dealing with reals). Therefore there exists a real number a such that $f(a) = b$ for every real b.

Some facts regarding surjectivity:

- Noticing a term of x not inside f(some variable) is pretty helpful. In Problem 3, RHS had a term of x not involving any f which helped us show the surjectivity of f.
- It is important to check the range and domain while proving whether a function is surjective or not.
- Surjectivity tells us that for all elements in the range, there is an element in the domain mapping to it. This is helpful in assigning arbitrary values to the function (e.g., assume $f(x) = 0$ if $f(x)$ is surjective and co-domain is the set of reals), if that helps to simplify the functional equation.

Definition. A function which is both injective and surjective is known as *bijective* function.

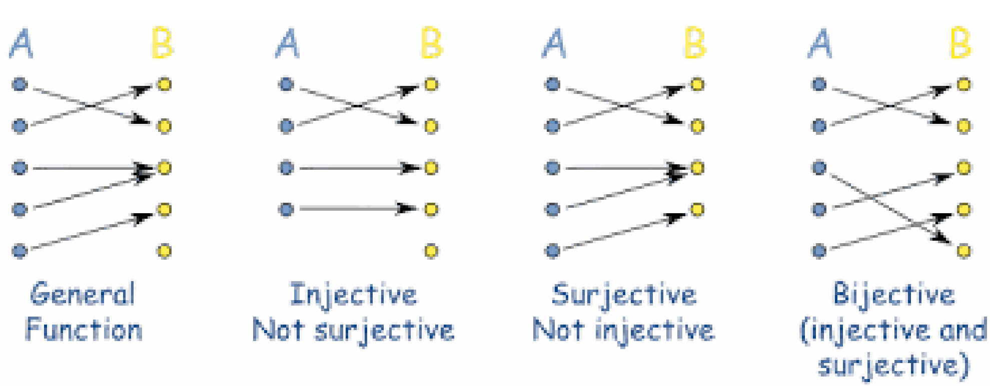


Figure 2. Injective, surjective and bijective function [2]

Monotonic. A monotonic function (or monotone function) is a function between ordered sets that preserves or reverses the given order. In order words, a monotonic function is an increasing or a decreasing function [1].

Definition. If $f: A \to B$ be a function and $f(x) \geq f(y)$ for all $x > y$ where x, y are in the set A, then the function is said to be increasing. If $f(x) > f(y)$ for all $x > y$ where x, y are in the set A, then the function is said to be strictly increasing.

Definition. If $f: A \to B$ be a function and $f(x) \leq f(y)$ for all $x > y$ where x, y are in the set A, then the function is said to be decreasing. If $f(x) < f(y)$ for all $x > y$ where x, y are in the set A, then the function is said to be strictly decreasing.

Let's try to recognize the monotonic property of functions.

- Problem 5: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = \sin x$. Is f monotonic?

  Observe that $\sin 90^\circ > \sin 30^\circ$ and $\sin 150^\circ < \sin 90^\circ$. From these relations, we realise that the sin function is not monotonic. The graph of the $\sin x$ function (Figure 3) illustrates this feature: the graph is rising in some regions and falling in other regions.

- Problem 6: Let $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = x^3$. Is f monotonic?

  We shall show that f is an increasing function. Suppose that $x > y$, then $f(x) - f(y) = x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. We have assumed that $x - y > 0$ and

$$x^2 + xy + y^2 = \frac{2x^2 + 2xy + 2y^2}{2} = \frac{(x^2 + y^2) + (x + y)^2}{2} \geq 0.$$

  Combining the results, we have $f(x) > f(y)$. The graph of the function (Figure 3) illustrates this property.

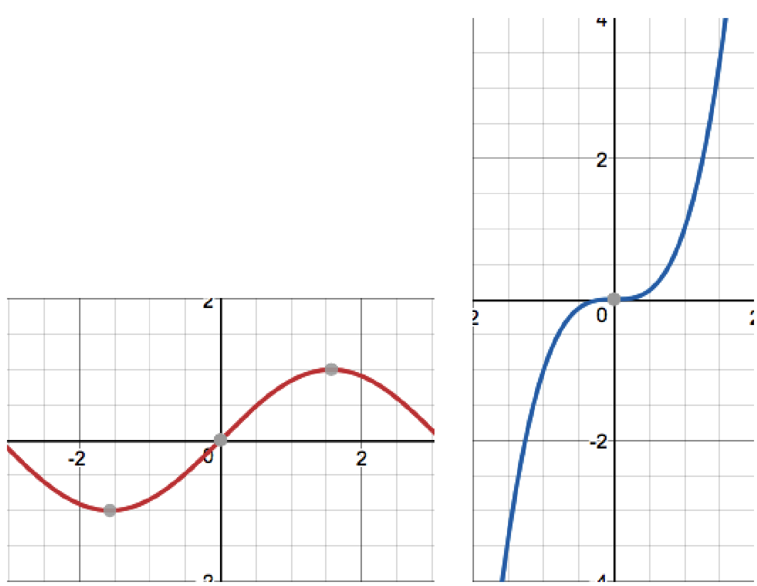


Figure 3. Graphs of $f(x) = \sin x$ and $f(x) = x^3$

Continuous. A function for which sufficiently small changes in the input result in arbitrarily small changes in the output is said to be *continuous*. Otherwise, it is said to be a *discontinuous* function [1].

Definition. Suppose $f(x)$ is a function. It is continuous at a point c if and only if the following conditions are satisfied:

- $f(c)$ exists.
- $\lim_{x\to c} f(x)$ exists.
- $\lim_{x\to c} f(x) = f(c)$.

In simplistic terms, the graph of a continuous function will be an uninterrupted curve in the domain where the function is defined. We won't go into the depths of continuity, as the concept involves ideas outside the scope of this article.

Practice problems
Determine whether the following functions are injective, surjective and/or monotonic.
(1) $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = c$ for some real constant c for all x.
(2) $f\colon \mathbb{C} \to \mathbb{C}$ satisfying $f(x) = x^3$ for all x.
(3) $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = \dfrac{x^2 + x}{2}$ for all x.
(4) $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying $f(x) = |x|$ for all odd x and $f(x) = -|x|$ for all even x.

Solving Functional Equations
Now that we know the basic properties of functions, let's discuss the ways to approach a functional equation.

- Guessing the solution: It may seem strange but an important technique to solve a FE is to guess the solution if feasible. However mere guessing and plugging in the original equation to prove the guess works is not sufficient since there may be other solutions also. Still, guessing a solution may help to simplify the functional equation in a manner from where the final solution can easily be obtained.
- Finding values of elements in domain: Finding values of $f(0), f(1)$, etc. depending on the domain is important. These values help to derive other relations which may lead to the solution.
- Determining properties of the functions: If we can prove that the unknown function is injective (or surjective/bijective/monotonic), it allows additional manipulation to the original FE which may lead to the solution.
- Deriving equations and manipulating it to get the solution: Most often, FE are equations in more than one variable. If so, then constraining the FE for suitable values of those variables can help to derive more properties of the function and may even solve it completely.

Example 1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(3x+2) = 5x$$

for all $x \in \mathbb{R}$.

Solution 1. In this problem, the equation given is similar to $f(x) =?$ Let's try to remove the $3x+2$ and replace it with a simpler term, say y. We want $3x+2 = y$ or $x = \frac{y-2}{3}$. Therefore plugging $x = \frac{y-2}{3}$, we get $f(y) = \frac{5(y-2)}{3}$ for all reals y ($3x+2$ covers all reals). The reasoning used shows that this function is the only one satisfying the given condition. Hence the solution is: $f(x) = \frac{5(x-2)}{3}$ for all $x \in \mathbb{R}$.

Example 2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x-y) = f(x) + f(y) - 2xy$$

for all $x, y \in \mathbb{R}$.

Solution 2. As discussed earlier, one way to start a FE is to find some values in the domain. In this problem the domain is $\mathbb{R}$. We can try to find $f(0)$.

Let $P(x,y)$ be the assertion of the problem statement. $P(0,0)$, i.e plugging $x = y = 0$ in the original equation gives $f(0) = f(0) + f(0) - 0 = 2f(0)$. This gives $f(0) = 0$.

Now that we have $f(0) = 0$, let's try to use this information. We may want to eliminate the LHS of the equation. $P(x,x)$ gives $f(0) = f(x) + f(x) - 2x^2 \implies f(x) = x^2$ for all x which is indeed a solution. The reasoning used shows that this function is the only one satisfying the given condition. Hence the solution is: $f(x) = x^2$ for all $x \in \mathbb{R}$.

Example 3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(2f(x) + f(y)) = 2x + f(y)$$

for all $x, y \in \mathbb{R}$.

Solution 3. One of the first steps in solving FE is to derive different properties of the unknown function like injectivity, surjectivity, etc. Once the property has been derived, this can then be applied to understand more about the function, sometimes even solving it completely.

In this problem, we notice an isolated x term in RHS. As indicated in section 2.1, this may help us to prove that the desired function is injective. Let's see how we can prove that f is injective.

Let $P(x,y)$ be the assertion of the problem statement, i.e $f(2f(x)+f(y)) = 2x + f(y)$.

Suppose $f(a) = f(b)$, $P(a,y)$ and $P(b,y)$ gives

$$2a + f(y) = f(2f(a) + f(y)) = f(2f(b) + f(y)) = 2b + f(y) \implies a = b.$$

So f is injective.

Let's now use this property to derive the value of $f(x)$. If we constrain the FE to $P(0,y)$ it gives $f(2f(0) + f(y)) = f(y)$. Since $f(x)$ is injective, this equation gives $2f(0) + f(y) = y \implies f(y) = y - 2f(0)$. Therefore $f(x) = x - 2f(0)$ for all $x \in \mathbb{R}$.

In order to find the value of $f(0)$, we put $x = 0$ into the relation $f(x) = x - 2f(0)$ and obtain $f(0) = -2f(0)$ or $f(0) = 0$.

Hence the solution is: $f(x) = x$ for all $x \in \mathbb{R}$.

Example 4. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f(f(n)) = n, f(f(n+2)+2) = n$$

for all $n \in \mathbb{Z}$ and $f(0) = 1$.

Solution 4. We are given $f(0) = 1$, let's try to use this information by using it in the equations. Let $P(n)$ be the assertion that $f(f(n)) = n$ and $Q(n)$ be the assertion that $f(f(n+2)+2) = n$. $P(0)$ gives $f(f(0)) = 0$ or $f(1) = 0$.

Observing the equations, we notice that we have a relation between $f(f(n))$, n and $f(f(n+2)+2)$ and n. We look to combine these conditions.

We have

$$f(f(n+2)+2) = n \implies f(f(f(n+2)+2)) = f(n) \implies f(n+2)+2 = f(n)$$

where the first equality is from $Q(n)$ and last one from $P(f(n+2)+2)$.

Let's summarise what we have got till now:

$$f(0) = 1, f(1) = 0, f(n+2)+2 = f(n).$$

It's easy to guess that $f(n) = 1 - n$ is a solution. We are working on integers, so induction is a good way to prove our claim. This part is left for the reader to prove.

Example 5. Find the value of $f(486)$ where $f\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function such that

$$f(f(n)) = 3n$$

for all $n \in \mathbb{N}$.

Solution 5. It is obvious that $f(n)$ is injective but it does not help us make any further progress about the value of $f(n)$. Therefore we need a different approach to solve this problem.

Let's see the values of the function for some initial values of the domain. Note this is quite useful to get a sense of how the function is behaving. Since the domain is $\mathbb{N}$, we will plug in values like $1, 2$, etc.

Let $P(n)$ be the assertion of the problem statement, i.e $f(f(n)) = 3n$. We have an additional condition that f is a strictly increasing function (recall definition 2.4), we'll try to use it.

$$P(1) \to f(f(1)) = 3.$$

We wish to find $f(1)$. If $f(1) = 1$, then $f(f(1)) = f(1) = 1 \neq 3$. If $f(1) \geq 3$, then from the strictly increasing condition,

$$3 = f(f(1)) > f(2) > f(1) \geq 3,$$

because $f(1) > 2$ (remember that we are assuming $f(1) \geq 3$). This means that $3 > 3$ which is contradictory. Therefore $f(1) = 2$ and $3 = f(f(1)) = f(2)$.

Now that we have $f(1)$ and $f(2)$, we look to utilise these facts. $P(2)$ gives

$$6 = f(f(2)) = f(3), 9 = f(f(3)) = f(6), 18 = f(f(6)) = f(9) \cdots.$$

Can we guess the pattern? It looks like $f(3^n) = 2 \cdot 3^n$ and $f(2 \cdot 3^n) = 3^{n+1}$ but we need a mathematical proof for it. Induction seems a plausible way to proceed since we are dealing with natural numbers.

Base case: $n = 1 \to f(3) = 6$ which we have already found out earlier and $f(6) = 9$ which we have also seen before. Inductive case: Hypothesis: $f(3^n) = 2 \cdot 3^n$ and $f(2 \cdot 3^n) = 3^{n+1}$.

$$2 \cdot 3^{n+1} = f(f(2 \cdot 3^n)) = f(3^{n+1})$$

and

$$3^{n+2} = f(f(3^{n+1})) = f(2 \cdot 3^{n+1}).$$

Hence $f(3^n) = 2 \cdot 3^n, f(2 \cdot 3^n) = 3^{n+1}$ for all natural n. Note that $486 = 2 \cdot 3^5$ so $f(486) = f(2 \cdot 3^5) = 3^6 = 729$.

The answer is: $f(486) = 729$.[1]

Example 6. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(-x) = -f(x), \quad f(x+1) = f(x) + 1, \quad f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$$

for all $x \in \mathbb{R}$ and $x \neq 0$.

Solution 6. There are no isolated terms which we could use to prove injectivity or surjectivity of $f(x)$. Also plugging values does not provide any idea about the behaviour of the function. Clearly we need to use a different approach to solve this problem.

We notice that there are some relations between $f(x)$ and $f(x+1), f(-x)$ and $f\left(\frac{1}{x}\right)$. Can we come up with a series starting from x and finally again reaching x using these possible transformations? In other words, we need a sequence of $x \to \ldots \to x$ using the moves $x \to x+1, x \to \frac{1}{x}, x \to -x$ for any x not equal to 0. Please see if you are able to find the cycle before looking at the cycle given below. Note that there could be many possible cycles.

$$x \to x+1 \to \frac{1}{x+1} \to \frac{-1}{x+1} \to 1 - \frac{1}{x+1} = \frac{x}{x+1} \to \frac{x+1}{x} = 1 + \frac{1}{x} \to \frac{1}{x} \to x.$$

Using the cycle, we get a relation in $f(x)$ which on simplifying gives $f(x) = 2x - f(x)$ for $x \neq 0, -1$. $\Rightarrow$ $f(x) = x$ for $x \neq 0, -1$. Lets' try to find $f(0), f(-1)$. We know $f(-1) = -f(1) = -1$ and $f(0) = -f(0) \implies f(0) = 0$.

The answer is: $f(x) = x$ for all $x \in \mathbb{R}$.

Conclusion

We have tried to show in this article how by repeatedly applying a few simple principles, we can make significant progress in understanding and solving functional equations. We will continue this theme in Part II of the article which will appear in the next issue. For now, here are a few sample problems which the reader could attempt on his/her own.

[1] $486 = 2 \cdot 3^5$, but what about 2001 which is neither a power of 3 nor twice a power of 3. Try to find the value of $f(2001)$. For the solution, refer my blog [3] at AoPS.

Example Problems

(1) (Korea 2000) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 - y^2) = (x - y)(f(x) + f(y))$$
for all $x, y \in \mathbb{R}$.

(2) Find all monotone functions $f\colon \mathbb{R} \to \mathbb{R}$ such that
$$f(4x) - f(3x) = 2x$$
for all $x \in \mathbb{R}$.

(3) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 + yf(z)) = xf(x) + zf(y)$$
for all $x, y, z \in \mathbb{R}$.

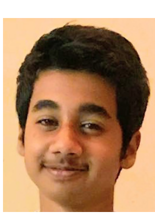
(4) Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f\left(\frac{f(x)}{y}\right) = yf(y)f(f(x))$$
for all $x, y \in \mathbb{R}^+$.

(5) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that
$$(x + y^2)f(yf(x)) = xyf(y^2 + f(x))$$
for all $x, y \in \mathbb{R}$.

(6) Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that
$$f(3x) - f(x) \leq 8x^2 + 2x, f(2x) - f(x) \geq 3x^2 + x$$
for all $x \in \mathbb{R}$.

References

1. https://en.wikipedia.org/wiki/Continuous_function
2. www.matematikaria.com/unit/injective-surjective-bijective.html
3. www.artofproblemsolving.com/community/c453393



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