The Constants of Mathematics

Part 1

SHAILESH SHIRALI

Science is full of constants. Probably the best known such constant is the velocity of light (c), made famous by Einstein's Special Theory of Relativity. (He postulated that all observers measuring the velocity of light in vacuum would obtain the same figure, regardless of their own velocity.) Other such constants, slightly less famous, are Planck's constant (h), the gravitational constant (G) which occurs in Newton's law of universal gravitation, the charge of the electron (e), the mass of the electron (m_e) and the mass of the proton (m_p). All these constants have *units* (so their values depend on the system of measurement), but there are also constants which are 'dimensionless'. For example, we have the 'fine-structure constant' α (also known as Sommerfeld's constant; it concerns the strength of the electromagnetic interaction between elementary charged particles) and constants like 3 (the number of independent dimensions of space) and 2 (which occurs as the exponent in so many force laws, e.g., Newton's universal law of gravitation).

In mathematics too, there are many constants. In one sense, of course, every number is a constant! But as in human society, in which all men are equal under the Constitution, yet some are "more equal than others" (apologies to George Orwell for this usage which is far removed from its original usage in *Animal Farm*), so too with numbers. Nature seems to have a particular love for some numbers, for they occur repeatedly in mathematical results, often in the most unexpected ways; numbers like π, e, γ and so on, and also numbers like 1 and 2.

Keywords: Constant, variable, irrational, commensurable

In this series, we make a whimsical journey visiting some well-known constants of mathematics; along the way we learn about their personalities, their peculiarities. In each case, we attempt to justify why mathematicians consider the number to be mathematically significant and hence is worthy of being called a 'constant'.

Pythagoras' constant: the square root of 2

We start with the number which has the honour of being the first one ever to be proved irrational: the square root of 2. It has the dubious honour of being the chief participant in the first great crisis in mathematics.

In what sense is $\sqrt{2}$ a mathematical constant? That is easy to see: the number is linked inextricably to the square, which is a fundamental geometric object. All squares are similar to one another, and the ratio of the diagonal to the side of any square is $\sqrt{2}$ (Figure 1). This follows from the theorem of Pythagoras, which explains the name given to the constant.

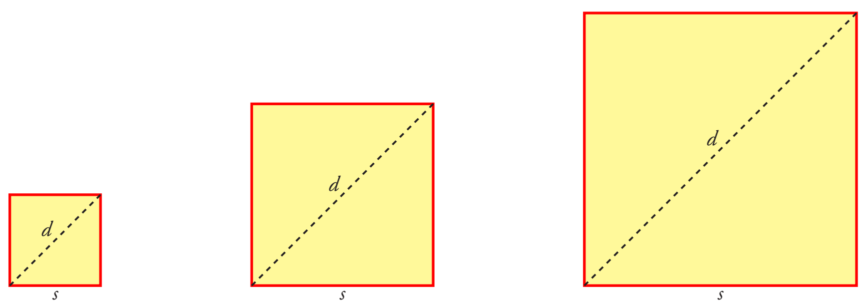

Figure 1. The ratio of diagonal to side, d/s, is the same for all squares: $d/s = \sqrt{2}$

As already stated, the constant $\sqrt{2}$ is notorious in being the first number shown to be irrational. This discovery was made by the school of Pythagoras (probably in the fifth century BCE), but it was expressed differently, thus: "The side and the diagonal of a square are not commensurable." (This means that no matter what unit of length we choose, it cannot fit a whole number of times into *both* the side *and* the diagonal.) A discovery of this kind if made today would be the source of much excitement and pleasure. But it appears that the discovery was not welcome to its discoverers! This will seem strange to us, but it has to be understood with reference to the Pythagorean world view, in which the role of the counting numbers was central. The word 'rational' may give us a clue to why this was so: nowadays, it is used to describe numbers that can be written as the ratio of two integers, but additionally it has the connotation of 'sane', 'orderly', 'logical', and so on. The fact that the same word is used to describe these two different attributes tells us that the Pythagorean view is still very much with us, so we continue to be Pythagoreans! This perspective may help us appreciate why the discovery that $\sqrt{2}$ is not commensurable provoked such a philosophic crisis, the first such in mathematics.

The Greeks were not the first to study the square root of 2. Earlier, the Babylonians studied it and had found some remarkable approximation schemes which are of interest to us even today.

Irrational nature of the square root of 2

We give several different proofs; each is (naturally) a "proof by contradiction". (Why 'naturally'? Because irrationality is essentially a negative concept; it asserts the *lack* of some characteristic, so there cannot be a direct proof of irrationality.)

Euclid's proof. This is the proof given in Euclid's *Elements*. It is perhaps the oldest formally articulated proof of any proposition in mathematics. It rests on two simple observations: (i) The square of an even integer is even. (ii) The square of an odd integer is odd. Here is the proof, expressed in modern algebraic language.

Suppose that $\sqrt{2} = a/b$ where a, b are positive integers. We may suppose that a, b are coprime, for if they do share a common factor, it can be 'canceled' from both the numbers, leaving the ratio a/b unchanged. But this means, in particular: a and b cannot *both* be even. By squaring the relation $\sqrt{2} = a/b$ we get:

$$2 = \frac{a^2}{b^2}, \quad \therefore \ a^2 = 2b^2, \tag{1}$$

from which follow these statements, in sequence: a^2 is even, hence a is even, hence $a = 2c$ for some positive integer c. These in turn lead to the following:

$$a^2 = 4c^2, \quad \therefore \ 2b^2 = 4c^2, \quad \therefore \ b^2 = 2c^2, \tag{2}$$

from which follow these statements, in sequence: b^2 is even, hence b is even. It thus transpires that both a and b are even. But this contradicts what we said above: that a and b cannot both be even. We conclude that the supposition made at the start has to be invalid, and hence that $\sqrt{2}$ is not rational. $\square$

A proof by descent. Since the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of positive integers is bounded below by 1, the following deduction is valid: *It is not possible to have an infinitely long, strictly decreasing sequence of positive integers.* (The two phrases 'infinitely long' and 'strictly decreasing' are crucial parts of this sentence.)

This may seem to be another of those 'obviously true' and trivial statements which cannot possibly yield anything of significance; but in fact many beautiful proofs are based on it. They are known collectively as **proofs by descent**. The proof we now offer, to show the irrationality of $\sqrt{2}$, is one such.

Suppose that a and b are positive integers such that $a/b = \sqrt{2}$. Then $b\sqrt{2} = a$. Using this property we define a set S as follows:

$$S = \text{the set of all positive integers } x \text{ such that } x\sqrt{2} \text{ is an integer.} \tag{3}$$

By definition, b lies in S; so S is non-empty. We shall now produce another positive integer which is smaller than b and lies in S.

The number we have in mind is $a - b$. First, we show that it has the desired property. It is certainly positive (for we have $a/b > 1$, hence $a > b$ and $a - b > 0$), and it is an integer, since a and b are integers. Now note that:

$$(a - b)\sqrt{2} = a\sqrt{2} - b\sqrt{2} = (b\sqrt{2}) \cdot \sqrt{2} - a = 2b - a. \tag{4}$$

Hence $a - b$ belongs to S. How can we be sure that $a - b$ is smaller than b? Let $b' = a - b$ and $a' = 2b - a$; then $a'/b' = \sqrt{2}$. Since a and b are integers, so are b' and a'. Since $a/b \approx 1.4$, it follows that $a > b$ but $a < 2b$, implying that $b' < b$. Hence $0 < b' < b$. We have thus found a positive integer b' which is smaller than b and lies in S.

This construction works with any integer in S. Thus we can find a positive integer b'' which is smaller than b' and lies in S. And so on.

We thus obtain an infinitely long, strictly decreasing sequence of positive integers. But such a sequence cannot exist!

Thus we have arrived at a contradiction. We conclude that the supposition made at the start (about the existence of the integers a and b) is invalid, and hence that $\sqrt{2}$ is not rational. $\square$

Another route to the above proof is the following. Observe that if $x^2 = 2$, then

$$2 - x = x^2 - x, \quad \therefore\ 2 - x = x(x-1). \tag{5}$$

Therefore, if $x = \sqrt{2}$ then:

$$x = \frac{2-x}{x-1}. \tag{6}$$

Now suppose that $\sqrt{2}$ is a rational number. Let $\sqrt{2} = a/b$ where a, b are positive integers. Substituting $x = a/b$ in (**??**) we see that

$$\frac{a}{b} = \frac{2 - a/b}{a/b - 1} = \frac{2b - a}{a - b}.$$

We have arrived at the same expression and the same numbers ($a - b$ and $2b - a$) as earlier.

Pictorial proof. This puts into an attractive, pictorial form the argument just presented. It starts with the supposition that $\sqrt{2} = a/b$ where a and b are positive integers.

- $AB = b$
- $AC = a$
- $AP = b$
- $PC = a - b$
- $BQ = a - b$
- $CQ = 2b - a$

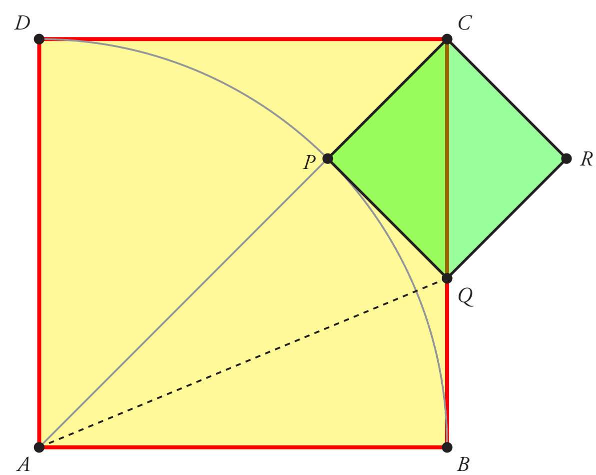

Figure 2.

Figure 2 displays a square $ABCD$ with side $AB = b$. Its diagonal AC has length $b\sqrt{2} = a$. By drawing an arc of a circle with radius b, centred at A, locate a point P on AC such that $AP = b$, and by drawing PQ perpendicular to PC, construct a square $CPQR$ on side CP, with Q on side CB. Join AQ. Since $AP = b$, we have $PC = a - b = PQ$.

Now consider $\triangle APQ$ and $\triangle ABQ$. They are RHS-congruent to each other, so $BQ = PQ$. It follows that $BQ = a - b$, and hence that $CQ = b - (a - b) = 2b - a$. Since a, b are integers, so are $a - b$ and $2b - a$. So the lengths of the side and diagonal of square $CPQR$ are positive integers.

Note what we have accomplished: starting with a square $ABCD$ whose side and diagonal have integer length, we have produced another square $CPQR$ whose side and diagonal also have integer length. Moreover, $CPQR$ is *strictly smaller* than $ABCD$. (Compare their diagonals: $CQ < CB$ and $CB < CA$, therefore $CQ < CA$.)

The same construction starting with square $CPQR$ will produce yet another square with integer side and diagonal, even smaller than square $CPQR$. The logic of the construction is such that we can continue this process forever. We thus get a shrinking sequence of integer-sided squares. This is clearly not possible — we cannot have indefinitely small, integer-sided squares! So we reach a contradiction, like earlier, and we conclude that we cannot construct such a configuration at all. Hence $\sqrt{2}$ is irrational. $\square$

Origami proof. The idea described above can be put in a pictorially attractive form in another way, using ideas from origami. Figure 3 (i) shows an isosceles right-angled $\triangle PQR$, right-angled at R. The bisector PS of $\measuredangle QPR$ has been marked. In Figure 3 (ii), the triangle has been folded along the angle bisector PS; what was originally $\triangle PSR$ has been folded upon $\triangle PST$, with side PR lying upon side PT.

Now suppose that $\sqrt{2}$ is a rational number, say $\sqrt{2} = a/b$ where a and b are positive integers. In Figure 3, choose the scale of the figure in such a manner that $PR = b$; then $PQ = a$. The sides of $\triangle PQR$ are b, b, a; these are all integers. Therefore, $\triangle PQR$ *is integer-sided, isosceles, and right-angled.* In Figure 3 (ii), $PT = PR$, hence $PT = b$ and $TQ = a - b$. In $\triangle TQS$, $\measuredangle TQS = 45° = \measuredangle TSQ$, hence $TS = TQ$, i.e., $TS = a - b$. Since $\triangle PSR \cong \triangle PST$, we get $SR = ST$, i.e., $SR = a - b$; therefore $QS = b - (a - b) = 2b - a$. So the sides of $\triangle TQS$ are $a - b, a - b, 2b - a$; these too are all integers. Therefore, $\triangle TQS$ *is integer-sided, isosceles, and right-angled.*

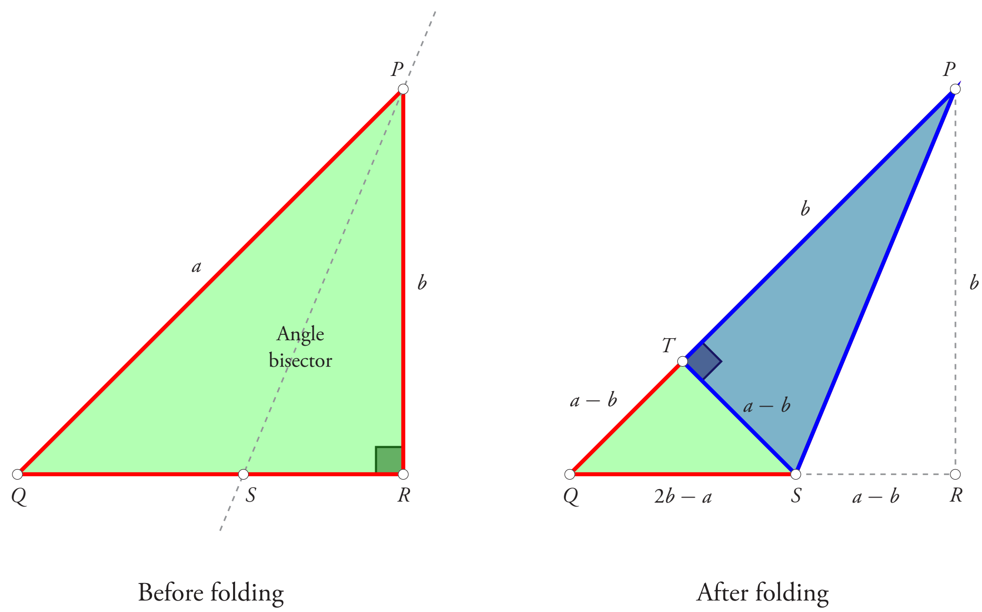

Figure 3.

Since $\triangle TQS$ lies within $\triangle PQR$, the sides of TQS are strictly smaller than the corresponding sides of PQR. Hence the existence of an integer-sided, isosceles, right-angled triangle has led to the existence of another such triangle but with strictly smaller sides. The very same construction applied to this smaller triangle will lead to the existence of yet another integer-sided, isosceles right-angled triangle.

This iterative step can be applied indefinitely, and we are forced to confront an infinite sequence of shrinking integer-sided triangles. This is clearly not possible; the condition that the sides are positive integers acts as an impassable barrier. Hence the initial assumption must be invalid; in other words, $\sqrt{2}$ cannot be a rational number. $\square$

Computing the square root of 2

To find good decimal approximations for $\sqrt{2}$ we may use the well-known "long-division method". But rather than traverse this well-trodden path, we shall use a different approach for approximating the square root of 2. It is very 'low-tech' in its requirements: all it needs is the expansion formula for $(a - b)^2$.

We start with the easily verified fact that $\sqrt{2}$ lies between 1 and 2, and hence that:

$$0 < \sqrt{2} - 1 < 1. \tag{7}$$

Let $\alpha = \sqrt{2} - 1$. Since α lies between 0 and 1, the same is true for the quantities α^2, α^3, α^4, …; that is, $0 < \alpha^n < 1$ for every positive integer n. Indeed, the larger the value of n, the closer the value of α^n to 0. This simple fact can be exploited to yield remarkably good approximations to α. Here's how we proceed. Squaring α using the binomial squaring formula, we get:

$$\alpha^2 = \left(\sqrt{2} - 1\right)^2 = 2 - 2\sqrt{2} + 1 = 3 - 2\sqrt{2}.$$

If we regard α^2 as a small quantity, i.e., $3 - 2\sqrt{2} \approx 0$, we get by division:

$$\sqrt{2} \approx \frac{3}{2}. \tag{8}$$

This is not a particularly good approximation, but it is noteworthy that we got it at all, and that too by such simple reasoning. We can improve it by continuing the squaring process. We get the following successively better approximations:

- $\alpha^4 = \left(3 - 2\sqrt{2}\right)^2 = 9 - 12\sqrt{2} + 8 = 17 - 12\sqrt{2}$, hence:

$$\sqrt{2} \approx \frac{17}{12}. \tag{9}$$

  This is much better!

- $\alpha^8 = \left(17 - 12\sqrt{2}\right)^2 = 289 - 408\sqrt{2} + 288 = 577 - 408\sqrt{2}$, hence:

$$\sqrt{2} \approx \frac{577}{408}. \tag{10}$$

  Even better ….

- $\alpha^{16} = \left(577 - 4082\sqrt{2}\right)^2 = 665857 - 470832\sqrt{2}$, hence:

$$\sqrt{2} \approx \frac{665857}{470832}. \tag{11}$$

- $\alpha^{32} = \left(665857 - 470832\sqrt{2}\right)^2 = 886731088897 - 627013566048\sqrt{2}$, hence:

$$\sqrt{2} \approx \frac{886731088897}{627013566048}. \tag{12}$$

It is worth examining how good these approximations are (each one necessarily yields an overestimate). Table 1 displays the results; each value may be compared with the actual value of $\sqrt{2}$ given in the last row. In just four steps, we have achieved close to twenty-five decimal place (d.p.) accuracy! That is indeed very impressive.

Remarks. Before closing this section we make two remarks.

- From any fraction a/b which is close to $\sqrt{2}$, in the sense that $|a - b\sqrt{2}|$ is a small quantity (close to 0; in any case, smaller than 1 in absolute value), we can obtain a better one by squaring, thus:

$$\left(a - b\sqrt{2}\right)^2 = \left(a^2 + 2b^2\right) - 2ab\sqrt{2}.$$

  Hence the new approximation is $\left(a^2 + 2b^2\right) \div 2ab$, which may be written as:

$$\frac{b}{a} + \frac{a}{2b}. \tag{13}$$

Number	Decimal expansion	Error
$\frac{17}{12}$	$1.41666\ \ldots$	2×10^{-3}
$\frac{577}{408}$	$1.41421\ 5686\ \ldots$	2×10^{-6}
$\frac{665857}{470832}$	$1.41421\ 35623\ 7468\ \ldots$	3×10^{-12}
$\frac{886731088897}{627013566048}$	$1.41421\ 35623\ 73095\ 04880\ 16896\ \ldots$	9×10^{-25}
$\sqrt{2}$	$1.41421\ 35623\ 73095\ 04880\ 16887\ \ldots$	

Table 1. Rational approximations to $\sqrt{2}$

Example: From the approximation $7/5 = 1.4$ (accurate to one d.p.) we get:

$$\frac{5}{7} + \frac{7}{10} = \frac{99}{70} \approx 1.41428,$$

which is accurate to 4 d.p. And from this we get:

$$\frac{70}{99} + \frac{99}{140} = \frac{19601}{13860} \approx 1.414213564,$$

which is accurate to 8 d.p. One more application yields 17 d.p. accuracy!

- The same logic can be used to get good rational approximations to numbers like $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$; indeed, the square root of any rational number. But it will not work for cube roots, fifth roots, and so on. (Why not?)
- Some of you may recognise in this scheme a low-tech version of the well-known Newton-Raphson scheme for numerically solving arbitrary single variable equations.

Sightings of the square root of 2

A4 Paper. Did you know that the familiar A4-sized sheet of paper we use in printers and photocopiers incorporates the magic number $\sqrt{2}$? The number $\sqrt{2}$ has the following property: $\sqrt{2} : 2 = 1 : \sqrt{2}$. Hence, if we take a rectangular sheet of paper whose length to width ratio is $\sqrt{2} : 1$ and fold it in half along its longer side, the folded sheet will have the same shape as the original one (it has the same length-to-width ratio). This is just the property that defines A4-sized paper! (For, if a rectangular sheet whose length to width ratio is $x : 1$ has such a property, then we must have $x/2 : 1 = 1 : x$. This equation has only one solution, $x = \sqrt{2}$, as we must have $x > 0$. So there is only one such ratio which 'works.') If we fold such a sheet in two, along the longer side, we get a A5-sized sheet, and if we fold *that* in two, we get a A6-sized sheet. Similarly we have A3-sized paper which would yield A4-size if folded in half. The length-to-width ratios are the same for all these sheets; namely, $\sqrt{2} : 1$. See Figure 4.

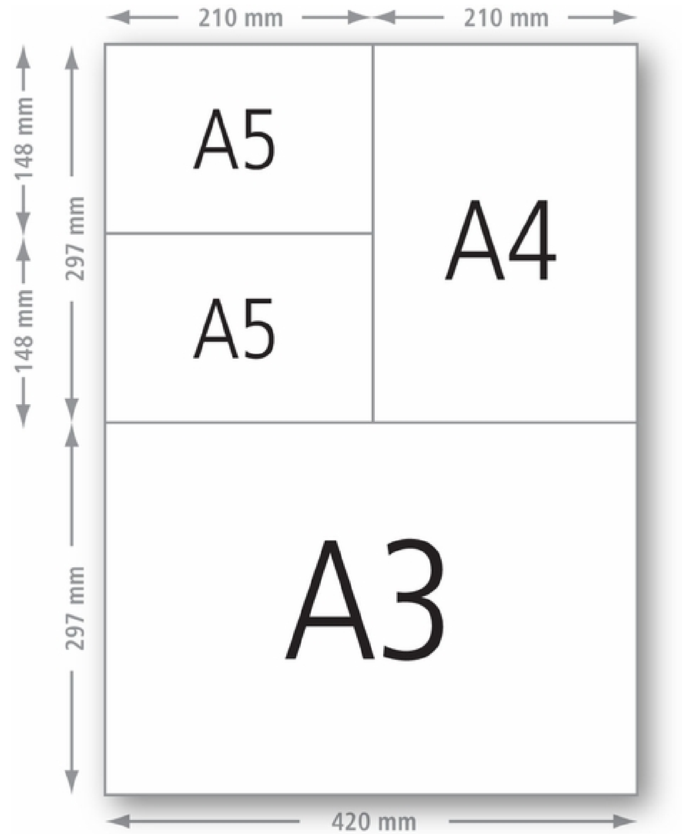

Figure 4. Paper sizes

…And a non-sighting of the square root of 2. The Boeing series of jet planes is well-known and their model numbers have become part of our everyday lexicon: Boeing 707, Boeing 747 and so on. Their very first model was the Boeing 707 and it has become part of folklore that it was so named because the angle between the wings and the body is 45° and, as is well-known, $\sin 45^\circ = 1/\sqrt{2} \approx 0.707$.

But this ancient wisdom has been debunked! In actual fact, the wingsweep angle of a Boeing 707 is 35°, not 45°. The actual reason behind the name is more pedestrian; see [1].

Figure 5. A Boeing 707; photo credit: https://www.boeing.com/history/products/707.page

To conclude, two beautiful formulas …

We conclude by displaying a couple of extremely beautiful expressions for the square root of 2.

A formula found by Euler (1707–1783). The first expression was found by the great Leonhard Euler. Try to prove it for yourself!

$$\sqrt{2} = \left(1+\frac{1}{3}\right) \times \left(1+\frac{1}{35}\right) \times \left(1+\frac{1}{99}\right) \times \left(1+\frac{1}{195}\right) \times \cdots . \tag{14}$$

The denominators in the fractions are

$$3 = 1\times 3, \quad 35 = 5\times 7, \quad 99 = 9\times 11, \quad 195 = 13\times 15, \quad \ldots .$$

A formula found by Francois Viète (1540–1603). The second expression is an amazing and beautiful formula connecting $\sqrt{2}$ and π:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\cdots \tag{15}$$

Try proving this for yourself. It is not too difficult! All you need is the following pair of results:

$$\sin 2x = 2\cdot \sin x\, \cos x, \qquad \cos x = \sqrt{\frac{1+\cos 2x}{2}}.$$

Closing remark. We have seen a few occurrences of $\sqrt{2}$ in this brief article. There are many, many more such sightings of this number in the world of mathematics but we shall leave the task of uncovering them to you.

References

1. Mike Lombardi, "Why 7's been a lucky number", https://www.boeing.com/news/frontiers/archive/2004/february/i_history.html

SHAILESH SHIRALI is the Director of Sahyadri School (KFI), Pune, and heads the Community Mathematics Centre based in Rishi Valley School (AP) and Sahyadri School KFI. He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.