problem corner

Triangle in a Rectangle

C⊗MαC

Problem. *In rectangle ABCD is inscribed triangle PBQ with P on side AD and Q on side CD. The areas of* $\triangle PAB$, $\triangle QBC$ *and* $\triangle DPQ$ *are p, q and r, respectively. Find the area s of* $\triangle PBQ$ *in terms of* p, q, r. (See Figure 1.)

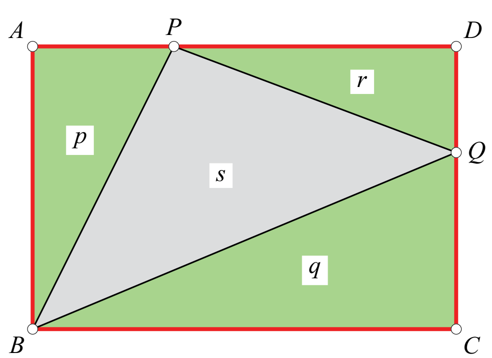

Figure 1.

Try to solve the problem on your own before looking up the solution!

Solution. Assign symbols for lengths as follows:

- $AB = a$
- $BC = b$
- $AP = x$
- $CQ = y$

We redraw Figure 1 with these symbols shown. It is remarkable that merely by defining these symbols and setting out the basic area relations, the answer can be found. Indeed, the "solution derives itself". This is surely one of those problems where the pen or pencil seems to possess its own intelligence.

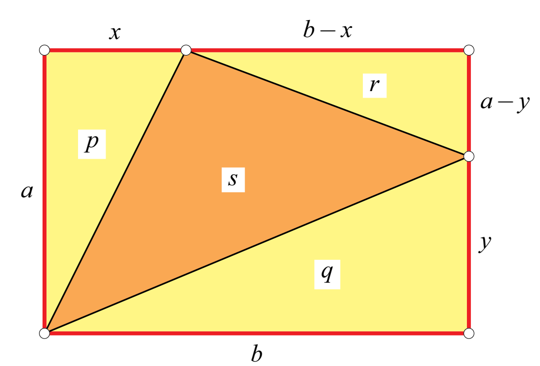

Figure 2.

We have the following relations:

$$ax = 2p,$$
$$by = 2q$$
$$(b-x)(a-y) = 2r.$$

From these we must find the value of ab. The third relation when expanded yields:

$$ab - ax - by + xy = 2r,$$
$$\therefore\ ab + xy = 2(p+q+r).$$

But we have:

$$abxy = 4pq, \quad \therefore\ xy = \frac{4pq}{ab}.$$

Hence:

$$ab + \frac{4pq}{ab} = 2(p+q+r).$$

This is clearly a quadratic equation in the unknown ab. Writing $z = ab$ we have:

$$z + \frac{4pq}{z} = 2(p+q+r),$$

$$\therefore\ z^2 - 2(p+q+r)z + 4pq = 0.$$

Solving this we get:

$$z = (p+q+r) \pm \sqrt{(p+q+r)^2 - 4pq}.$$

The sign ambiguity must be resolved. But z surely must exceed $p+q+r$. Hence:

$$z = (p+q+r) + \sqrt{(p+q+r)^2 - 4pq}.$$

The area s of $\triangle PBQ$ is therefore given by:

$$s = \sqrt{(p+q+r)^2 - 4pq}.$$

This relation may be expressed in more aesthetically pleasing ways, e.g.:

$$(p+q+r)^2 - s^2 = 4pq.$$

As might have been expected, the expression for s is symmetric in p and q. (Why?)

Yet another proof for the "Sum of an Arithmetic Progression" formula

by: Faiz Imam

This proof is based on an article by Martin Griffiths; see [1]. We wish to find the sum S_n of the first n terms of an arithmetic progression (AP),

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \cdots + (a + [n-1]d),$$

where a and d are respectively the first term and common difference of the AP. If $d = 0$ then clearly $S_n = na$. Now assume that $d \neq 0$. Let $c = a/d$, so $a = cd$. Therefore:

$$\begin{aligned} S_n &= cd + (cd+d) + (cd+2d) + (cd+3d) + \cdots + (cd + [n-1]d) \\ &= dc + [(c+1) + (c+2) + (c+3) + \cdots + (c + (n-1))]d. \qquad (1) \end{aligned}$$

Now we treat c as an integer. Using the formula for the sum of the first k positive integers, we evaluate the sum in line (1):

$$\frac{(c+n-1)(c+n)}{2} - \frac{(c-1)c}{2} = \frac{n(2c+n-1)}{2} \qquad \text{(on simplification).}$$

Hence:

$$\begin{aligned} S_n = d \times \frac{n(2c+n-1)}{2} &= \frac{n}{2} \times [2cd + (n-1)d] \\ &= \frac{n}{2} \times [2a + (n-1)d]. \end{aligned}$$

We have got the correct formula for the sum of an AP though we treated c as an integer which it clearly may not be. How is this possible?

REFERENCES

[1] Martin Griffiths, Sums of powers of the terms in any finite Arithmetic Progression, The Mathematical Gazette, Vol. 86, No. 506 (Jul., 2002), pp. 269-271