problem corner

Problems for the Senior School

Problem Editors : PRITHWIJIT DE & SHAILESH SHIRALI

Problems for Solution

The problems in this set are adapted from the Romanian Mathematical Competitions, 2014.

Problem IV-1-S.1
Let $A = \{1, 3, 3^2, 3^3, \dots, 3^{2014}\}$. A *partition* of A is a union of non-empty disjoint subsets of A.

(a) Prove that there is no partition of A such that the product of all the elements in each subset is a square.
(b) Does there exist a partition of A such that the sum of elements in each subset is a square?

Problem IV-1-S.2
Let ABC be a triangle in which $\measuredangle A = 135^\circ$. The perpendicular to line AB at A intersects side BC at D, and the bisector of $\measuredangle B$ intersects side AC at E. Find the measure of $\measuredangle BED$ (see Figure 1).

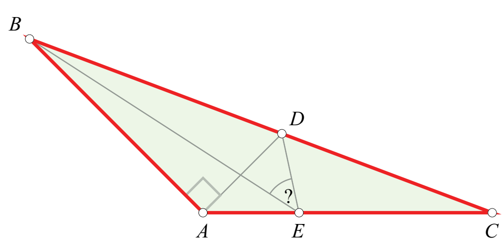

Figure 1.

Problem IV-1-S.3
Determine all pairs (n, p) of positive integers such that

$$\left(n^2+1\right)\left(p^2+1\right)+45 = 2(2n+1)(3p+1).$$

Problem IV-1-S.4
Determine all irrational numbers x such that both $x^2 + x$ and $x^3 + 2x^2$ are integers.

Problem IV-1-S.5
Find all pairs (p, q) of prime numbers, with $p \le q$, such that

$$p(2q+1) + q(2p+1) = 2\left(p^2+q^2\right).$$

Solutions of Problems in Issue-III-3 (November 2014)

Solution to problem III-3-S.1 *If* $(x-y+z)^2 = x^2-y^2+z^2$, *prove: either* $x=y$ *or* $z=y$.

First observe that $x^2-y^2+z^2 = (x+z)^2-y^2-2zx = (x-y+z)(x+y+z)-2zx$. Hence if $(x-y+z)^2 = x^2-y^2+z^2$, then

$$\begin{aligned} 2zx &= (x-y+z)(x+y+z)-(x-y+z)^2 \\ &= (x-y+z)(2y). \end{aligned}$$

This yields:

$$y^2-(z+x)y+zx=0, \qquad \therefore (y-x)(y-z)=0.$$

Hence $x=y$ or $z=y$.

Solution to problem III-3-S.2 *Prove that the numbers* 10017, 100117, 1001117, ... *are all divisible by* 53.

Let a_n be the n-th number in the given sequence. Then $a_1 = 53\times 189$. Also: $a_n = 10a_{n-1}-53$ for each n. Hence if a_{n-1} is a multiple of 53, so is a_n. Since a_1 is a multiple of 53, it follows by the principle of induction that a_n is a multiple of 53 for every n.

Solution to problem III-3-S.3 *Let* $ABCD$ *be a parallelogram. Let the bisector of* $\measuredangle ABD$ *meet* CD *produced at* X *and let the bisector of* $\measuredangle CBD$ *meet* AD *produced at* Y. *Prove that the bisector of* $\measuredangle ABC$ *is perpendicular to* XY.

Let $\measuredangle ABD = 2x$ and $\measuredangle CBD = 2y$ (see Figure 2). Then $\measuredangle BXD = x$ and $\measuredangle BYD = y$. Thus in triangle BDX, $BD = DX$ and in triangle BDY, $BD = DY$. Thus $DX = DY$; and since $\measuredangle XDY = \measuredangle ADC = 2(x+y)$, it follows that $\measuredangle DXY = \measuredangle DYX = 90^\circ-(x+y)$. Hence $\measuredangle BXY = \measuredangle BXD + \measuredangle DXY = 90^\circ - y$.

If the bisector of $\measuredangle ABC$ meets XY at Z then in triangle XBZ, $\measuredangle XBZ + \measuredangle BXZ = (x+y-x)+90^\circ-y = 90^\circ$. Hence $\measuredangle BZX = 90^\circ$.

Solution to problem III-3-S.4 *Prove that if* $a_1 \le a_2 \le a_3 \le \ldots \le a_{10}$, *then*

$$\frac{a_1+\ldots+a_6}{6} \le \frac{a_1+\ldots+a_{10}}{10}.$$

Observe that

$$\begin{aligned} &6(a_1+\cdots+a_{10})-10(a_1+\cdots+a_6) = \\ &\quad 6(a_7+\cdots+a_{10})-4(a_1+\cdots+a_6). \end{aligned}$$

Now $6(a_7+\cdots+a_{10}) \ge 24a_7$ and $4(a_1+\cdots+a_6) \le 24a_6$. Hence:

$$6(a_1+\cdots+a_{10})-10(a_1+\cdots+a_6) \ge 24(a_7-a_6) \ge 0.$$

The result follows.

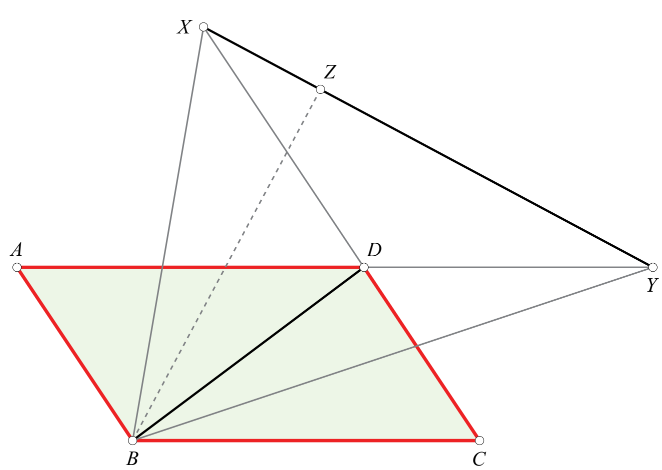

Figure 2.