Problems for the Middle School

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problem corner

Problems for Solution

Problem IV-1-M.1
If the sum of the reciprocals of three non-zero real numbers is zero, is it possible that the sum of the three numbers is zero?

Problem IV-1-M.2
If a and b are integers such that $a + 2b$ and $b + 2a$ are square numbers, show that each of a and b is divisible by 3.

Problem IV-1-M.3
Show that a power of 2 cannot be represented as a sum of two or more consecutive positive integers.

Problem IV-1-M.4
In $\triangle ABC$, one of the mid-segments is longer than one of its medians. Show that $\triangle ABC$ is obtuse-angled. (Note: The mid-segment of a triangle is a segment joining the midpoints of two sides of a triangle.)

Problem IV-1-M.5
Show that in any circle, two non-diametrical chords cannot both bisect each other.

Problem IV-1-M.6
A and B are two boxes. Box A contains 100 white marbles, while box B contains 100 black marbles. We take 10 marbles at random from box A and put them into box B. After this we take out 10 marbles at random from box B and put them in box A. Which is now larger: the number of black marbles in box A, or the number of white marbles in box B?

Problem IV-1-M.7
Let $a_1, a_2, a_3, \dots, a_n$ represent the numbers $1, 2, 3, \dots, n$ subjected to an arbitrary arrangement. Assume that n is odd. Consider the number

$$X = (a_1 - 1)(a_2 - 2)(a_3 - 3) \dots (a_n - n).$$

What can be said about the parity of X. Is this number even or odd?

Solutions of Problems in Issue-III-3 (November 2014)

Solution to problem III-2-M.1 *Amar, Basil, Celia and Dharam are four children. Basil's age is greater than twice Amar's age; the sum of Amar's and Celia's ages is less than Basil's age. Dharam is older than Basil. If Celia is* 6 *years old, and Dharam is* 9 *years old, find Basil's age. (All ages are in whole numbers).*

Let us represent the names by their first letters A, B, C, D. Then $C = 6, D = 9, A + C < B, B < D$. Hence $A + 6 < 9$, i.e., $A < 3$, which yields $A = 1$ or 2. Also, $B < 9$.

Next, $A + C < B$ also gives $B - A > C$ which means $B - A > 6$. Now:

- If $A = 2$, then we get $8 < B < 9$ which is not possible.
- If $A = 1$, then we get $7 < B < 9$ giving $B = 8$. So the required answer is 8 years.

Solution to problem III-2-M.2 *Mary's teacher notes the test scores of* 32 *students in her class. She finds that the median score is* 80 *and the range of the scores is* 40*. The teacher then tells the class that their average score is* 58*. Mary contends that her teacher has gone wrong somewhere. Who is right, Mary or her teacher?* [*Fryer Contest, 2003*]

The average of the class (as claimed by the teacher) is 58. Therefore the total mark is $32 \times 58 = 1856$.

Since the median mark is 80, at least 16 students will have a mark of at least 80; hence the total mark of these 16 students will be at least $16 \times 80 = 1280$.

This implies that the total mark of the remaining 16 students will be at most $1856 - 1280 = 576$; but these 16 students should have actually scored at least $16 \times 40 = 640$. This is so because as the median is 80 and the range is 40, there are students who got at least 80 and so the lowest possible individual score is 40 (we have been told that the difference between the highest and lowest marks is 40).

This contradiction shows that Mary is justified in her suspicion. The teacher has indeed made an error in the computation.

Solution to problem III-2-M.3 *Select* 50 *distinct integers from the first* 100 *natural numbers, such that their sum is* 2900*. What is the least possible number of even integers amongst these?*

To avoid even integers as far as possible, let us try to get the sum 2900 using only the odd integers. The sum of the first 50 odd integers $1, 3, 5, 7, \ldots, 99$ is $50^2 = 2500$. So we need 400 more to get the sum 2900. For this, we have to use only the even integers, keeping intact the total number of integers as 50.

To the extent possible, we try to exchange the smallest odd integers for the largest even integers, in pairs, since 400 is even. So:

- We replace $\{1, 3\}$ with $\{100, 98\}$. This makes the sum $2500 - 1 - 3 + 100 + 98 = 2694$.
- As this is not sufficient we replace $\{5, 7\}$ with $\{96, 94\}$. Now the sum becomes $2694 - 5 - 7 + 96 + 94 = 2872$.
- We are still short by 28, so we try one more exchange. We replace $\{9, 11\}$ with $\{20, 28\}$, and this works: $2872 - 9 - 11 + 20 + 28 = 2900$.

We have used only 6 even integers, namely: $100, 98, 96, 94, 28, 20$. This is the minimal number of even integers needed.

Solution to problem III-2-M.4 *Find the digits A and B if the product $2AA \times 3B5$ is a multiple of* 12*.*

The given product is a multiple of 12, which means it is divisible by 3 and 4. The factor of 4 must come from the term $2AA$ since $3B5$ is odd. Invoking the test for divisibility by 4, we see that $A = 0$, 4 or 8.

- If $A = 0$, then $2AA$ is not divisible by 3, so $3B5$ must be divisible by 3, hence $B = 1, 4$ or 7.
- If $A = 4$, then $2AA$ is not divisible by 3, so $3B5$ must be divisible by 3, hence $B = 1, 4$ or 7.
- If $A = 8$, then $2AA$ is divisible by 3, so the B in $3B5$ can be any digit.

So if $A = 0$ or 4 then $B \in \{1, 4, 7\}$, and if $A = 8$ then B can be any digit. So there are $3 + 3 + 10 = 16$ possible combinations which satisfy the conditions of the problem.

Solution to problem III-2-M.5 *One altitude of a triangle is tangent to its circumcircle. Prove that some angle of the triangle has measure larger than* 90° *but less than* 135°*.*

Let the triangle be ABC. Suppose that AD, the altitude to side BC through A, is tangent to the circumcircle at A. As the tangent lies outside the circle, it intersects BC at a point D on its extension. Assume that D lies beyond B on ray $\overrightarrow{CB}$. Hence $\measuredangle ABC > 90^\circ$. Consider $\triangle ADB$; it is right-angled at D. From the intersecting chords theorem we know that $DB \times DC = DA^2$. (See below for the theorem statement.) We also have $DB < DC$. Hence $DB < DA$. Hence $\measuredangle ABD > \measuredangle BAD$, and so $\measuredangle ABD > 45^\circ$. It follows that $\measuredangle ABC < 135^\circ$. Therefore $\measuredangle ABC$ lies between 90° and 135°. (See Figure 1.)

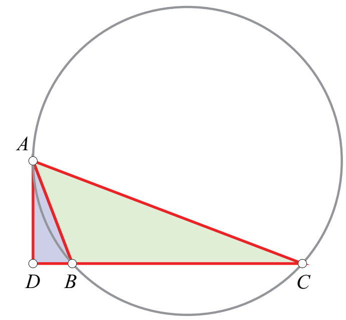

Figure 1.

Note: The **intersecting chords theorem** states the following. Through a point P let two lines l and m be drawn intersecting a given circle Γ at pairs of points $\{A, B\}$ and $\{C, D\}$ respectively. Then $PA \times PB = PC \times PD$. The point P could lie either inside or outside the circle, or on it. If P lies outside the circle and one of the lines, say m, is tangent to the circle at C, then the statement yields: $PA \times PB = PC^2$.

The Versatile 9

It seems that these three 9's have learned the art of cooperatively working together!