feature

Lurking within any triangle…

Morley's Miracle – Part II

…is an equilateral triangle

This article continues the series started in the last issue, wherein we study one of the most celebrated and beautiful theorems of Euclidean geometry: Morley's Miracle. In this segment we examine some approaches based on trigonometry.

V G Tikekar

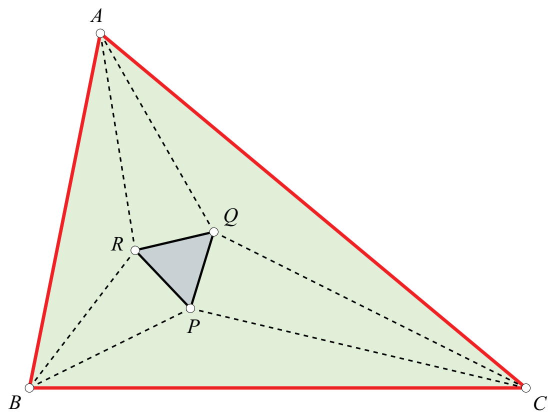


Figure 1. Morley's theorem: The angle trisectors closest to each side intersect in points which are the vertices of an equilateral triangle

In Part I of this article we had narrated the history of this theorem and discussed a beautiful 'pure geometry' proof found by M. T. Naraniengar over a century back. Readers will recall the curious logic used: the proof *starts* with an equilateral triangle and then constructs a configuration similar to the original one, and reaches the desired conclusion this way. (We remarked at that point that many of the pure geometry proofs known today proceed in this way. In Part III of this article, we will show another such proof.)

Keywords: *Angle trisector, equilateral triangle, supplementary angle, addition formula, triple angle formula, sine rule, cosine rule*

In contrast, the trigonometric proof is straightforward: it establishes that the triangle in question is equilateral simply by computing the lengths of its sides and checking that they are equal. In that sense it is very direct, and far from subtle. However, the algebraic steps are challenging! (Not, you might say, for the faint of heart.)

We start by summarizing (without proof) some facts we need from trigonometry. Proofs will be found in the textbooks used for classes 11–12. We use the usual notation: the sides of $\triangle ABC$ are a, b, c; the angles are A, B, C (with side a opposite $\angle A$ and so on); the radius of the circumcircle is R, and the radius of the incircle is r.

Supplementary angles identity: For any angle x,

$$\sin(180^\circ - x) = \sin x. \qquad (1)$$

Addition formula: For any two angles x and y,

$$\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y. \qquad (2)$$

Triple angle formula: For any angle x,

$$\sin 3x = 3\sin x - 4\sin^3 x. \qquad (3)$$

Triple angle formula (product form): For any angle x,

$$\sin 3x = 4\sin x \cdot \sin(60^\circ - x) \cdot \sin(60^\circ + x). \qquad (4)$$

Sine rule: In $\triangle ABC$, the following identity holds:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \qquad (5)$$

Cosine rule: In $\triangle ABC$, the following identity holds:

$$a^2 = b^2 + c^2 - 2bc\ \cos A, \qquad (6)$$

with similar relations for sides b and c.

Of these, perhaps the only one which may look unfamiliar is (4): the product form of the triple angle formula. It is a nice exercise to prove it on one's own.

Now for the details of the trigonometric proof. Let $\angle A = 3x$, $\angle B = 3y$ and $\angle C = 3z$. Then the three angles created by the trisectors at vertex A are x each, the three angles created by the trisectors at vertex B are y each, and the three angles created by the trisectors at vertex C are z each. (See Figure 2.) Our strategy from this point on is straightforward:

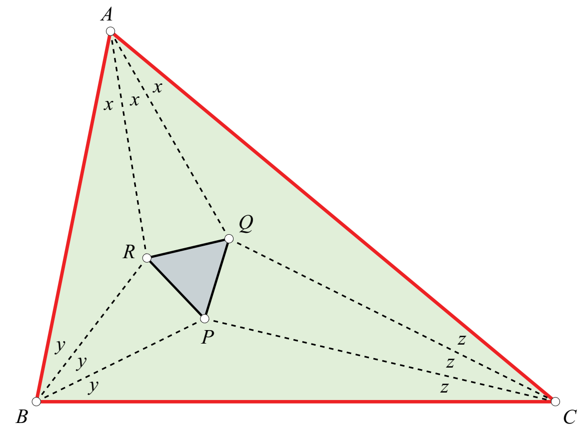


Figure 2.

(i) We use the sine rule in $\triangle PBC$ and $\triangle ARB$ to compute the lengths of BP and BR.

(ii) We transform the two expressions so obtained (for the lengths of BP and BR) using the triple angle formula (quoted above).

(iii) Armed with these expressions for the lengths of BP and BR, we use the cosine rule in $\triangle BPR$ and thus find the length of PR.

Steps (i) and (ii) are easy, but (iii) involves a substantial amount of manipulative algebra.

Now let us get started. Without any loss of generality we take the circumcircle of $\triangle ABC$ to have unit radius (i.e., $R = 1$). The sine rule applied in $\triangle PBC$ yields:

$$\frac{BP}{\sin z} = \frac{BC}{\sin \angle BPC} = \frac{2R\ \sin 3x}{\sin(180^\circ - y - z)} = \frac{2R\ \sin 3x}{\sin(y+z)} = \frac{2\sin 3x}{\sin(60^\circ - x)},$$

since $R = 1$ and $x + y + z = 60^\circ$. Hence:

$$BP = \frac{2\sin z \cdot \sin 3x}{\sin(60^\circ - x)}.$$

In the same way we get:

$$BR = \frac{2\sin x \cdot \sin 3z}{\sin(60^\circ - z)}.$$

Using the triple angle formula (product form), we transform these to the following:

$$BP = \frac{2\sin z \cdot 4 \cdot \sin(60^\circ - x) \cdot \sin x \cdot \sin(60^\circ + x)}{\sin(60^\circ - x)}$$

$$= 8\sin z \cdot \sin x \cdot \sin(60^\circ + x).$$

Similarly, $BR = 8\sin x \cdot \sin z \cdot \sin(60^\circ + z)$.

Now we apply the cosine rule to $\triangle BPR$, using the above expressions:

$$\begin{aligned}
PR^2 &= BP^2 + BR^2 - 2BP \cdot BR \cdot \cos y \\
&= 64 \sin^2 z \cdot \sin^2 x \cdot \sin^2(60^\circ + x) \\
&\qquad + 64 \sin^2 x \cdot \sin^2 z \cdot \sin^2(60^\circ + z) \\
&\qquad - 128 \sin^2 z \cdot \sin^2 x \cdot \sin(60^\circ + x) \\
&\qquad \cdot \sin(60^\circ + z) \cdot \cos y \\
&= 64 \sin^2 x \cdot \sin^2 z \cdot \left[\sin^2(60^\circ + x)\right. \\
&\qquad + \sin^2(60^\circ + z) - 2\sin(60^\circ + x) \\
&\qquad \left.\cdot \sin(60^\circ + z) \cdot \cos y\right].
\end{aligned}$$

In the last line, note the angles occurring in the expression within square brackets: $60^\circ + x$, $60^\circ + z$ and y. Their sum is $120^\circ + (x + y + z) = 180^\circ$. Hence there exists a triangle with angles $60^\circ + x$, $60^\circ + z$ and y. The form of the expression within the square brackets now invites the next step.

Consider such a triangle UVW (see Figure 3) and apply the sine rule to it. We get:

$$\frac{UW}{\sin y} = \frac{UV}{\sin(60^\circ + z)} = \frac{VW}{\sin(60^\circ + x)} = 2k,$$

where k is the radius of the circumcircle of $\triangle UVW$. Now we apply the cosine rule to the same triangle. We get:

$$\begin{aligned}
UW^2 &= UV^2 + VW^2 - 2\,UV \cdot VW \cdot \cos y, \\
\therefore\ 4k^2 \sin^2 y &= 4k^2 \sin^2(60^\circ + z) \\
&\qquad + 4k^2 \sin^2(60^\circ + x) \\
&\qquad - 8k^2 \sin(60^\circ + z) \\
&\qquad \cdot \sin(60^\circ + x) \cos y, \\
\therefore\ \sin^2 y &= \sin^2(60^\circ + z) + \sin^2(60^\circ + x) \\
&\qquad - 2\sin(60^\circ + z) \\
&\qquad \cdot \sin(60^\circ + x) \cos y.
\end{aligned}$$

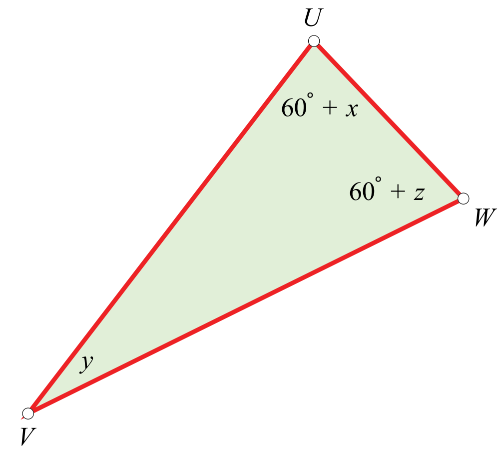


Figure 3. Triangle UVW with angles $60^\circ + x$, $60^\circ + z$ and y

This is an identity connecting any three angles x, y, z whose sum is 60°.

Going back to the expression we had found for PR^2 we find an amazing simplification:

$$PR^2 = 64 \sin^2 x \cdot \sin^2 y \cdot \sin^2 z,$$

and therefore:

$$PR = 8 \sin x \cdot \sin y \cdot \sin z.$$

What a lovely formula!

It is immediately obvious from the form of the above expression that we do not need to do any further computations. For, the expression obtained is completely symmetric in x, y, z (it does not 'prefer' any of x, y, z to the other two quantities), and this tells us that we will get exactly the same expression for PQ as well as QR. Hence $PQ = QR = PR$, and it follows that $\triangle PQR$ is equilateral.

References

[1] Bankoff, L. "A simple proof of the Morley theorem", Math. Mag., **35** (1962) 223–224

[2] Bogomolny, A. "Morley's Miracle", http://www.cut-the-knot.org/triangle/Morley/index.shtml.

[3] Child, J. M. "Proof of Morley's Theorem". Math. Gaz. **11**, 171, 1923.

[4] Conway, J. H. "Proof of Morley's Theorem", http://www.cambridge2000.com/memos/pdf/conway2.pdf

[5] Coxeter, H. S. M. "Introduction to Geometry", 2nd ed., Wiley, New York, 1969

[6] Coxeter, H. S. M. & Greitzer, S. L. "Geometry Revisited", Random House/Singer, New York, 1967

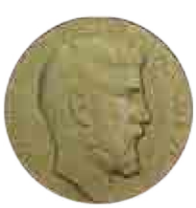
[7] Gardner, M. "New Mathematical Diversions from Scientific American", Simon and Schuster, New York, 1966, pp. 198, 206

[8] Honsberger, R. "Morley's Theorem", Mathematical Gems I. Washington, DC: Math. Assoc. Amer., pp. 92-98, 1973.
[9] Johnson, R. A. Advanced Euclidean Geometry, Dover, New York, 1960
[10] Knight, G. H. Morley's theorem, New Zealand Math. Mag., **13** (1976) 5-8.
[11] Weisstein, Eric W. "Morley's Theorem". From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/MorleysTheorem.html.
[12] Naraniengar, M. T. Solution to Morley's problem, "The Educational Times, with many Papers and Solutions in addition to those published in The Educational Times," New Series, **15** (1909) 47
[13] Robson, A. Morley's theorem, Math. Gaz., **11** (1922-1923) 310-311
[14] Stonebridge, B. A Simple Geometric Proof of Morley's Trisector Theorem, http://ms.appliedprobability.org/data/files/selected articles/42-1-1.pdf
[15] Wikipedia. "Morley's trisector theorem". http://en.wikipedia.org/wiki/Morley's_trisector_theorem.

PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1994. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.

QUOTES FROM MANJUL BHARGAVA

The Canadian-American number theorist Manjul Bhargava was awarded the Fields Medal at the ICM held in July 2014, in Seoul. Here are some quotes from an interview he gave recently to Mr. ChidanandaRajaghatta of Times of India.



[On his being awarded the Fields Medal.]
I am honored to be a recipient of the Fields Medal; beyond that, it is a source of encouragement and inspiration, and I hope that it is so also for my students and collaborators and colleagues who work with me.

[On whether math genius is a product of meticulous hard work and practice]
While a good memory and a copious supply of talent [are] very helpful, there is no substitute for hard work. Of course, this hard work has to be done in a way where one is always making progress, and where one is approaching this work with realistic short and long term goals with a global vision for what on is trying to achieve.

[On whether Ramanujan's genius was of the same kind as that of the character portrayed by Robin Williams in the movie Good Will Hunting]
Good Will Hunting was great as a movie, but as you might imagine, in reality most mathematics is not done the way it was portrayed in the movie. It requires years of hard work put in to get something out. Ramanujan was a talent of a level that has never been seen, but he certainly put in the hours as well to get the results he was interested in.