problem corner

Some problems from the Olympiads

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In this edition of "Adventures" we consider some problems from various mathematics contests. The first two are from the very first International Mathematical Olympiad (IMO), held in 1959 in Romania. For those familiar with the IMO in recent years, it may come as a shock to see the vast difference in level between the early years and present times. One longs for the good old days! For those not familiar with the IMO, here is some information about its structure. There are a total of six problems in the Olympiad, administered over two days, three on each day. It is customary for the first and fourth problems to be the easiest in the collection, and for the third and sixth ones to be the most difficult.

In addition we study some problems (numbers 3 and 4 below) from the extensive problem collection available at the following website: http://web.archive.org/web/20040405065644/http://www.kalva.demon.co.uk/index.html. Specifically we use this page: http://web.archive.org/web/20040530211115/http://www.kalva.demon.co.uk/aime/aime83.html.

We state the problems first so you have a chance to try them out on your own.

(1) **Problem 1 of IMO 1959.** *Show that the fraction* $\dfrac{21n+4}{14n+3}$ *is irreducible for every natural number n.*

Note: An "irreducible" fraction is one which is in its simplest form. For example, for $n = 1, 2$ and 3, the given fraction takes the values $\frac{25}{17}$, $\frac{46}{31}$ and $\frac{67}{45}$ respectively. Each of these is in its simplest form.

(2) **Problem 4 of IMO 1959.** *Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.*

(3) *What is the largest prime factor of the central binomial coefficient* $\binom{2000}{1000}$*?* Note: Another notation for $\binom{2000}{1000}$ is ${}^{2000}C_{1000}$.

(4) *How many four-digit numbers with first digit* 2 *have exactly two identical digits?* Note: We refer to numbers like 2001 or 2012.

Discussion and solutions

Problem 1 of IMO 1959. We are asked to show that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n. As noted, the term 'irreducible' means that the fraction cannot be simplified further and is already in its simplest form; no cancellation of factors can be done between numerator and denominator. For example the fractions $\frac{5}{3}$ and $\frac{15}{8}$ are irreducible, but not $\frac{10}{15}$ which can be 'reduced' (simplified) to $\frac{2}{3}$. Hence another way of stating the problem is: *Show that the numbers* $21n + 14$ *and* $14n + 3$ *have no common factors for every natural number* n. Here is yet another way of stating the problem: *Show that the numbers* $21n + 14$ *and* $14n + 3$ *are coprime for every natural number* n.

The word "coprime" may suggest that we will need to find the prime factors of the two numbers involved ($21n + 14$ and $14n + 3$) and then check that there is no overlap in the two sets of primes. However this is an extremely difficult problem! Indeed, if n is some unspecified number, there is no way whatever of finding the prime factors of either $21n + 14$ or $14n + 3$.

Fortunately there is another approach. It rests on a simple fact: *Pairs of consecutive numbers are coprime*. (For example: 9 and 10 are coprime, though both of them are composite numbers. Similarly, 20 and 21 are coprime, as are 25 and 26.) And this in turn rests on another simple fact: *If d is a divisor of two integers a and b, then d is a divisor of $a - b$*. Indeed, if $a = md$ and $b = nd$, where m and n are integers, then $a - b = (m - n)d$. Hence if the consecutive integers n and $n + 1$ share a common divisor d, then d must be a divisor of $(n + 1) - n$, i.e., d must be a divisor of 1, which forces d to be *equal* to 1. It follows that n and $n + 1$ can share no factor other than 1, i.e., they are coprime.

A moment's thought shows that this result can be extended: *If integers a and b have multiples ma and nb which differ by* 1, *then a and b are coprime*. For, if the integer d is a common divisor of a and b, then d is a common divisor of ma and nb, hence d is a divisor of $ma - nb$, i.e., d is a divisor of 1, hence $d = 1$. So a and b are coprime.

So our task reduces to find multiples of $21n + 4$ and $14n + 3$ which differ by 1. But the relevant multipliers are easily found, using the fact that $21 : 14 = 3 : 2$. We get:

$$2(21n + 4) - 3(14n + 3) = 8 - 9 = -1.$$

So $2(21n + 4)$ and $3(14n + 3)$ are consecutive integers, and it follows that $21n + 4$ and $14n + 3$ are coprime, as required.

Problem 4 of IMO 1959. We are asked to construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle. Here, "construct" means: work out a procedure using ruler-and-compass which will accomplish the stated end.

The way we shall solve this is to find out the ratio of the legs using algebra, then to draw a triangle which has the right 'shape' (i.e., it is similar to the desired triangle), and finally to construct the desired triangle.

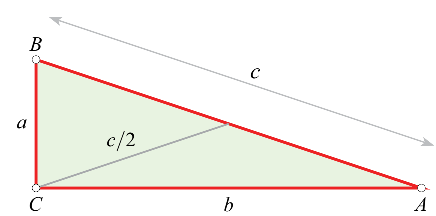

Figure 1.

Let the triangle be as depicted in Figure 1, with legs a and b, and hypotenuse c. Then the median to the hypotenuse has length $c/2$. (Do you see why? Remember that this is a right-angled triangle, so its circumcentre coincides with the midpoint of the hypotenuse.) The condition stated in the problem yields: $ab = (c/2)^2$, i.e., $c^2 = 4ab$. We also have: $a^2 + b^2 = c^2$. The two conditions yield: $a^2 - 4ab + b^2 = 0$. Treating this as a quadratic equation in b we get:

$$b = \frac{4a \pm \sqrt{16a^2 - 4a^2}}{2} = (2 \pm \sqrt{3})a.$$

Hence $b/a = 2 + \sqrt{3}$ or $b/a = 2 - \sqrt{3}$. Let us consider the first possibility. Draw a segment $B'C'$ with length 1, as shown in Figure 2, and a line ℓ perpendicular to $B'C'$ at C'. Draw a ray at B' making an angle of 60° to ray $B'C'$ and let it meet ℓ at D'. Then $C'D'$ has length $\sqrt{3}$. Locate point A' further along ℓ such that $D'A'$ has length 2. Then

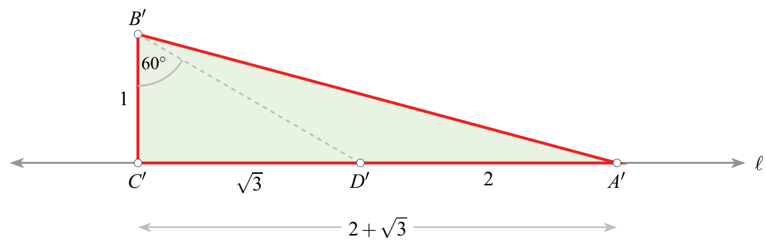

Figure 2.

$C'A'$ has length $2+\sqrt{3}$, hence the right triangle $A'B'C'$ has the right shape.

Having constructed a triangle with the desired shape, we know the angles of the desired triangle. The rest is easy. On a segment AB with the given length c (which will serve as the hypotenuse), we mark off at its two ends the appropriate angles from $\triangle A'B'C'$, using the standard procedure for transferring angles. (We leave the details to you to complete.) This will give us the desired $\triangle ABC$.

You may wonder what happens with the second possibility, $b/a = 2-\sqrt{3}$. We leave it to you to work out the answer, but a strong hint comes from the fact that $2-\sqrt{3}$ is the reciprocal of $2+\sqrt{3}$. So the ratio b/a is the reciprocal of the ratio we got the previous time. What does this tell you about the shape of the corresponding triangle?

Largest prime factor of a central binomial coefficient. We are asked to find the largest prime factor of the binomial coefficient $\binom{2000}{1000}$. By definition we have:

$$\begin{aligned}\binom{2000}{1000} &= \frac{2000!}{1000!\ 1000!}\\ &= \frac{2000}{1} \times \frac{1999}{2} \times \frac{1998}{3} \times \cdots \times \frac{1001}{1000}.\end{aligned}$$

It should be clear from this expression that every prime number between 1000 and 2000 is a divisor of the binomial coefficient $\binom{2000}{1000}$; for there is nothing in the denominator that can cancel such a prime. So an easy strategy to answer this question is to find the largest prime number between 1000 and 2000. We start from 2000 (i.e., with the largest number in the range) and work our way downwards. We quickly find that 1999 is a prime number. Hence 1999 is the answer to our question.

Counting four-digit numbers. We are asked to count the number of four-digit numbers with first digit 2 having exactly two identical digits (numbers like 2001 or 2012). We subdivide the set of such numbers into two categories: those for which the repeated digit is 2, and for which the repeated digit is different from 2.

If the repeated digit is 2, then the second, third and fourth digits are of the form $2, x, y$ where $\{x, y\}$ is a two element subset of the digit set $\{0, 1, 3, 4, 5, 6, 7, 8, 9\}$. The number of such subsets is $\binom{9}{2} = 9 \times 8/2 = 36$. These three digits can be permuted in $3! = 6$ ways. Hence the number of numbers in this category is $36 \times 6 = 216$.

If the repeated digit is different from 2, then the second, third and fourth digits are of the form x, x, y (in some order) where x, y are not equal to 2. We can select x in 9 possible ways, and having selected x, we can select y in 8 possible ways. The digits can be permuted in $3!/2 = 3$ ways. Hence the number of numbers in this category is $9 \times 8 \times 3 = 216$.

Therefore the total number of numbers of the stated type is $216 + 216 = 432$. This is the required answer.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.