

# Learning Mathematics through Geometrical Inquiry

*The possibility of learning mathematics through ICT support has never before been so promising. With the free software GeoGebra on your computer, you can conduct mathematical investigations in a time frame that people some generations back never could have dreamed of. Since the GeoGebra environment allows you to do calculations, geometrical constructions, measurements, and modelling in the same file, you can explore mathematics in a wonderful way. I am particularly fond of geometrical inquiry based problems that can open up other areas of mathematics.*

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## Introduction

Traditionally, the use of tools such as compass, protractor and ruler has been a common practice in learning geometry in school classrooms. However, in recent years, a class of software tools known as Dynamic Geometry Software (DGS), has revolutionised the teaching and learning of geometry and has brought about a shift of paradigm in the way concepts in geometry can be made accessible to students.

Some of the DGS popularly used in schools across the world include GeoGebra (Hohenwater, 2001), CabriTM (Laborde and Bellemain, 1993) and Geometer's SketchpadTM (Jacjw, 1991). One of the greatest affordances of a Dynamic Geometry Environment (DGE) is that it allows the user to drag parts of a geometrical figure in a geometry window, while the measurements of the figure change dynamically in an algebra window. Thus geometrical figures are dynamic and not static as they are when drawn on paper. The dynamic nature of a DGE figure enables the user to observe properties about it which

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remain invariant and those which do not. Since a DGE also provides measurements of different parts of the figure, such as the lengths of sides and angle measures, it enables the user to make conjectures by observing the changes or invariance in the measurements. In fact, in a DGE, the underlying principle is “to provide a family of diagrams as representing a set of geometrical objects and relations instead of a single static diagram”. Researchers have also described a DGE as a ‘microworld’ which provides rich opportunities for students to make and test conjectures. Thus a DGE provides a environment for performing investigations and for working on inquiry based learning activities and the mathematics education community has strongly emphasised the same (Brown &Walter, 2005; Da Ponte, 2007; Jones & Shaw, 1988; Leikin, 2004;Silver, 1994; Wells, 1999, 2001).

In this article we shall describe the exploration of a theorem in geometry which can provide opportunities for performing investigations, observing patterns and making conjectures. The DGS used as the vehicle for exploration is GeoGebra. It is free and open-source and may be downloaded from [www.geogebra.org](http://www.geogebra.org).

Personally, my view of geometrical investigations changed over twenty years ago, when I first read about Walter’s theorem and the investigations done by a grade 9 student, Ryan Morgan. Walter’s theorem was first presented as Marion Walter’s theorem in the November 1993 issue of the *Mathematics Teacher* in the section called Readers Reflections (Cuoco, Goldenberg, and Mark 1993). In short it says the following, illustrated by figure 1.

Marion Walter’s theorem appeared again in the May 1996 issue of the *Mathematics Teacher* in an article called Morgan’s theorem (Watanabe, Hanson, and Nowosielski 1996). The article tells the story about young Ryan Morgan, a ninth grader with good mathematical sense and a strong desire to explore a problem to its limits. Ryan’s mathematics teacher Frank Nowosielski presented Walter’s theorem to his class in the fall of 1993 and asked them to explore if it would hold for various types of triangles. Ryan was not satisfied merely with verifying Walter’s theorem. He was interested in finding out what would

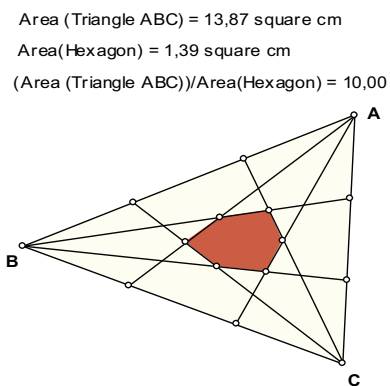


Figure 1. Illustrating Walter’s theorem: If the points of trisection of the sides of a triangle are joined to the opposite vertices, the hexagon formed inside the triangle has one-tenth the area of the triangle

happen if the sides of the triangles were divided into more than three congruent segments. Ryan and his teacher called the process of dividing a side of a triangle into  $n$  congruent segments “ $n$ -secting”. Using the Geometer’s Sketchpad, Ryan experimented with different  $n$ -sections (Watanabe et. al., p. 420). The article explains the methodology of Ryan’s investigations and what it led to; not surprisingly, Ryan was invited to present his conjecture at a special mathematics colloquium at Towson State University in 1994. If you are interested in how pre-service mathematics teachers handled the challenges of this problem, read about it in Lingefjord & Holmquist (2003).

A configuration similar to Walter’s Theorem, but not equivalent to it, comes from the following exploration. Observe the two equilateral triangles in Figure 2. The side of the larger triangle has been trisected in order to construct the inner triangle, which is formed when we join each vertex of the triangle to one of the trisecting vertices on the opposite side, in a cyclically symmetric way. One finds now that the ratio of the areas of the two triangles is exactly 1 : 7. The reader may verify this using GeoGebra.

### Investigation 1

Measure the same ratio if you divide the side of the outer triangle into 5 equal sections and construct the smallest possible equilateral triangle inside. (This happens if you join each vertex of the original triangle to the vertex on the opposite side

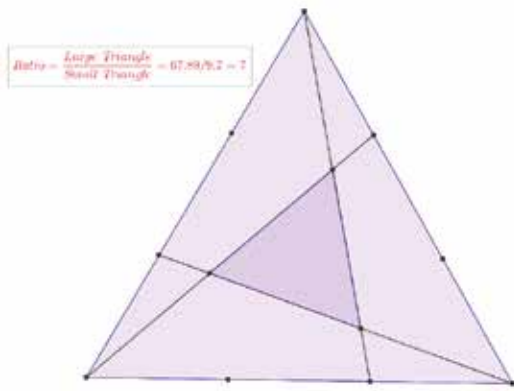


Figure 2. GeoGebra figure showing that the ratio of areas of the inner and outer triangles is 1:7

which is closest to the midpoint of that side, in a cyclically symmetric way.) Repeat the procedure for 7, 9, 11 and 13 equal sections. Do you see any pattern emerging in the sequence of ratios?

The activity is easy to start with, but soon presents an interesting challenge. Firstly, we need a method for dividing a line segment into  $n$  equal parts, a process which we refer to as  $n$ -secting. This is easy to accomplish using the available tools of GeoGebra (which are more powerful than the tools of the Geometry instrument box). The code for finding the ratio of areas is given at the end of the article.

Figure 3 (a) and 3 (b) show the output in GeoGebra of dividing a side of the equilateral triangle into 5 equal parts and 7 equal parts respectively.

## Investigation 2

Do you see any pattern in the sequence of ratios of the areas of the inner and outer triangles in the various cases of ' $n$ -secting' the sides of the outer triangle? From the cases of  $n = 3, 5$  and  $7$ , can you make a conjecture regarding the ratio for  $n = 9$  or  $11$ ?

From our constructions we find the following. For a 3-section the ratio of the areas is 1 : 7. For a 5-section and a 7-section the ratios are 1 : 19 and 1 : 37 respectively. We may also say that the ratio for a 1-section is 1 : 1 (this is true in a rather trivial sense). The numbers 1, 7, 19, 37 form a sequence in which the differences between the first three pairs of consecutive numbers are 6, 12, 18. Can we conclude that the next increment will be 24 and

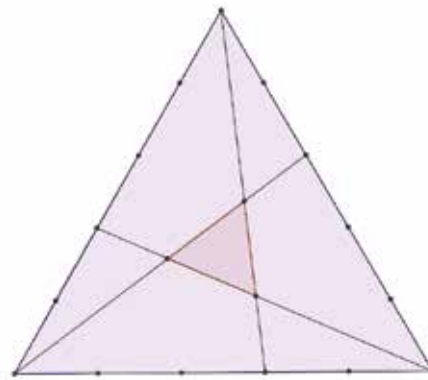


Figure 3(a)

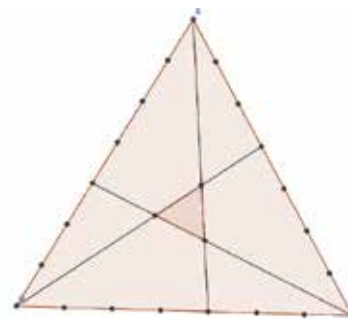


Figure 3(b)

Figure 3. GeoGebra figures showing the inner equilateral triangles formed after (a) 5-secting, (b) 7-secting the sides of the outer equilateral triangle. The reader may compute the ratio of the areas of the inner and outer triangles using GeoGebra.

the ratio will be 1 : 61 (since  $37 + 24 = 61$ )? Let us find out. See Figure 4. A GeoGebra investigation shows that the ratio is indeed 1 : 61 for the case  $n = 9$ . Thus we have experimentally verified our conjecture for the next term.

The ratios lead us to this sequence: 1, 7, 19, 37, 61. An attempt to find a pattern in this sequence reveals the following:

$$\begin{aligned}
 a_0 &= 1, \\
 a_1 &= 7 = 1 + 1 \cdot 6, \\
 a_2 &= 19 = 1 + 2 \cdot 6 + 1 \cdot 6, \\
 a_3 &= 37 = 1 + 3 \cdot 6 + 2 \cdot 6 + 1 \cdot 6, \\
 a_4 &= 61 = 1 + 4 \cdot 6 + 3 \cdot 6 + 2 \cdot 6 + 1 \cdot 6.
 \end{aligned}$$

Generalising this pattern for the  $k$ -th term we get  $a_k = 1 + k \cdot 6 + (k - 1) \cdot 6 + (k - 2) \cdot 6 + \dots + 2 \cdot 6 + 1 \cdot 6$ .

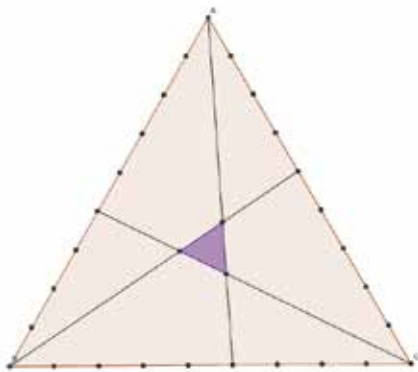


Figure 4. GeoGebra finds the ratio of the areas of the inner and outer triangle to be 1:61

Using the formula for the sum  $1 + 2 + 3 + \dots + k$ , we get:

$$a_k = 1 + 6 \frac{k(k+1)}{2} = 3k^2 + 3k + 1.$$

The observed pattern therefore suggests that the ratio between the areas of the triangles is governed by the formula  $a_k = 3k^2 + 3k + 1$ . But note here that the  $k$  in this formula is **not** the same as  $n$ , the number of parts into which the sides of the triangle were divided. The values of  $n$  corresponding to  $k = 1, 2, 3, 4, \dots$  are  $n = 3, 5, 7, 9, \dots$  from which we see that  $n = 2k + 1$ . This yields  $k = (n - 1)/2$ . Substituting for  $k$  into the above relation, we find that the formula for the ratio of areas in terms of  $n$  is  $(3n^2 + 1)/4$ . We may check that by substituting  $n = 3, 5, 7, 9$  into this formula, we get the numbers 7, 19, 37, 61.

But maybe there is more to find out about the numbers 1, 7, 19, 37, 61, ...?

A web search for the numbers 1, 7, 19, 37, 61, ... reveals that they are called **hexagonal centred**

**numbers** and they are obtained from counting the number of spots making up a full hexagon.

Let us now explore the sequence 1, 7, 19, 37, ...graphically, by creating ordered pairs of points  $(n, t(n))$ , where  $n$  represents the number of the term and  $t(n)$  the  $n$ -th term of the sequence. The first three points are:  $(0, 1), (1, 7), (2, 19)$ . Three points not in a line are enough to fix a quadratic curve. When we fit a quadratic function through the points we get a graph shown in Figure 6. Note that the function  $f(x) = 3x^2 + 3x + 1$  is the same as the formula obtained by us when we generalised our sequence algebraically.

Now when we have this model, we can go back to Figure 2 and make some more observations.

If we label the length of the sides of the outer triangle as  $s$  and the length of the sides in the inner triangle as  $x$ , then the distance from one of the inner triangles' vertex to the closest vertex of the large triangle is  $kx$ , where  $k$  is a positive number whose value has to be found. Now the segments  $AP, CQ, BR$  have equal length. In triangle  $AOC$ ,  $AC = s, OC = kx, AO = (k + 1)x$  and angle  $AOC = 120$  degrees (which is so because angle  $AOR$  equals 60 degrees). Hence by the cosine rule

$$AC^2 = AO^2 + OC^2 - 2AO \cdot OC \cdot \cos AOC.$$

This leads to:

$$s^2 = (k + 1)^2 x^2 + k^2 x^2 + 2k(k + 1)x^2/2, \text{ i.e.,}$$

$$\left(\frac{s}{x}\right)^2 = 3k^2 + 3k + 1.$$

This equation has a link to the formula GeoGebra helped us find. Note that  $k$  indicates a specific ratio between the two triangles but it is **not** the same as  $n$ , the number of parts into which we divided the sides of the outer triangle. The task of

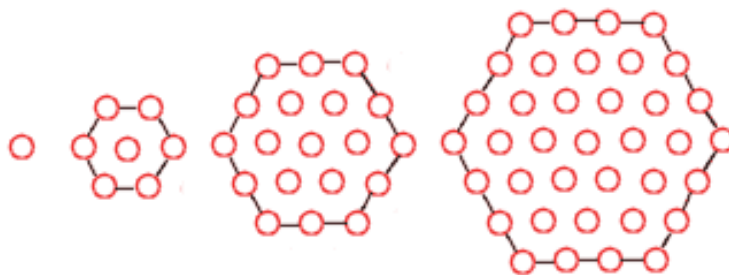


Figure 5. Hexagonal centred numbers (see <http://www.drking.org.uk/hexagons/misc/numbers.html>)

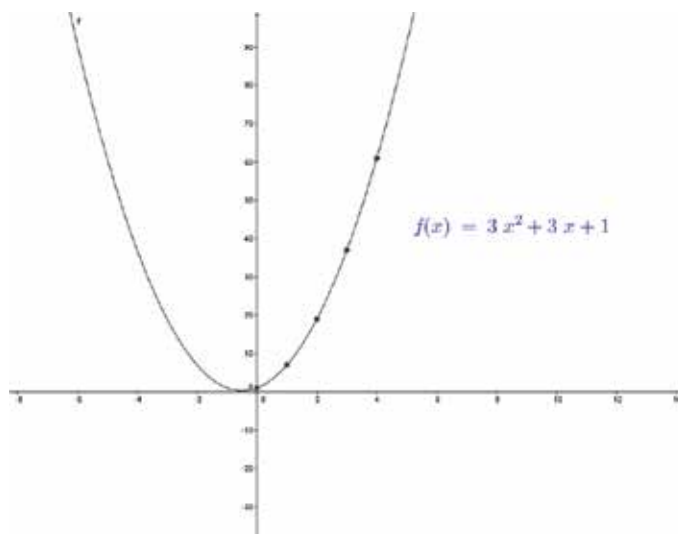


Figure 6. Modelling tools within GeoGebra

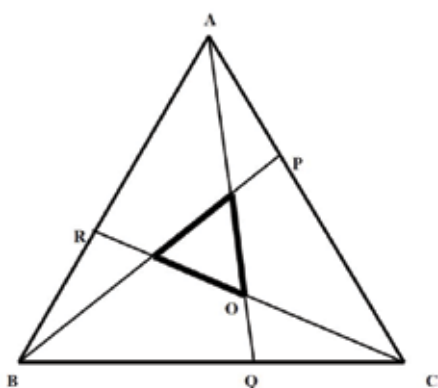


Figure 7. Using the cosine rule to compute some ratios

theoretically proving the formula  $(3n^2 + 1)/4$  for the ratio of the areas of the triangles remains to be done, but we leave this for the reader.

This exploratory activity presents us with an opportunity to explore a problem both

geometrically and algebraically. Pictorial representations on the computer screen (using GeoGebra) along with numerical and graphical representations of the sequence of ratios of areas of the equilateral triangles provide an opportunity to generalise the problem and create a link between the geometrical exploration and algebraic formulation.

### Closing remarks

Some mathematics teachers see the potential of using dynamic geometry to explore mathematical concepts and to provide inquiry-based learning experiences to students. Others cite improvements in the classroom atmosphere, increase in motivation levels of students and the efficiency of showing many examples at once as some of the reasons for incorporating dynamic geometry in their classrooms (Lampert1993; Ruthven, et al. 2005).

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### Appendix: Procedure for carrying out the investigation using GeoGebra

We give the steps in pseudo-code. The GeoGebra macro can be prepared using this code.

1. Select three non-collinear points A, B, C. Define triangle 'abc' with vertices A, B, C. GeoGebra command: `abc = Polygon[A, B, C]`.
2. Let  $n = 3$ .
3. Construct a slider for n. Let n move in increments of 2 (it takes values 3, 5, 7, ...).
4. Let  $k = (n-1)/(2n)$ .
5. Define:  $D = B + k*(C-B)$ ,  $E = C + k*(A-C)$ ,  $F = A + k*(B-A)$ .
6. Define: `ad = Segment[A, D]`, `be = Segment[B, E]`, `cf = Segment[C, F]`.
7. Define: `P = Intersect[ad, be]`, `Q = Intersect[be, cf]`, `R = Intersect[cf, ad]`.
8. Define triangle 'pqr' with vertices P, Q, R. GeoGebra command: `pqr = Polygon[P, Q, R]`.
9. Compute the ratio of areas,  $abc/pqr$ . GeoGebra command: `ratio = abc/pqr`.
10. Study dependence of 'ratio' on n. Also, for fixed n, study the dependence of 'ratio' on the choice of the triangle, by dragging the vertices about.



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