Ingenuity in Algebra

Gems from Bhāskarāchārya II

Number Problems from Ancient India

feature

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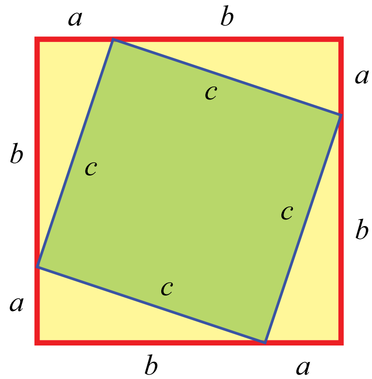
Bhāskarā II, pre-eminent mathematician-astronomer of ancient India, was born in 1114 AD in Bijapur (present-day Karnataka). Thus, the present year marks his nine hundredth birth anniversary. Events have been organized across the country in honour of this event.

Bhāskarā served as head of an astronomical observatory located at Ujjain (in present-day Madhya Pradesh). He is best known for his work *Līlāvatī* ('The Beautiful'). This book occurs as part of a larger work, *Siddhānta Siromani* ('Crown of Treatises'), which also has an important section on advanced algebra, *Bījagaṇita* ('Seed Arithmetic'), and two works on astronomy, *Grahagaṇita*, on the motions of planets, and *Golādhyāya* ('The Sphere'). This work was written when he was 36 years old.

Līlāvatī is a work on basic mathematics, covering arithmetic, simple algebra, geometry and simple mensuration. The famous 'Behold!' proof-without-words of the Pythagorean theorem (see Figure 1) is from this book.

There is a lovely legend behind *Līlāvatī* which is worth quoting. The legend goes that Bhāskarā had a daughter named Līlāvatī. She came of age, her wedding was arranged, and the long-awaited day finally came. The young Līlāvatī, ever curious about things around her, peeped into the water clock



$$(a+b)^2 = c^2 + \left(4 \times \tfrac{1}{2}ab\right)$$

$$\therefore \; a^2 + b^2 = c^2$$

Figure 1. Proof of the Pythagorean theorem. No words needed …

being used to ensure that events took place at the auspicious hour. As she did so, a pearl from her necklace fell into the water, blocked one of the channels, and so interrupted the flow. This upset the calculations, and the wedding took place at the wrong time, with the result that Līlāvatī was widowed at an early age. To console her, Bhāskarā composed his masterpiece, *Līlāvatī*, ... Did it really happen this way? Of course, we do not know, and we have no way of knowing. We do not even know if Bhāskarā had a daughter by that name!

Bījagaṇita is mainly about the extraction of roots of equations of different kinds; its level of exposition is higher than that of *Līlāvatī*. Interestingly, some of the material of *Līlāvatī* is dealt with again in *Bījagaṇita*, but the analysis is deeper, and algorithms are not presented merely in cook-book fashion. It covers topics such as:

- The rules of operations for working with zero and infinity (Bhāskarā II is the first mathematician to state explicitly and correctly the rules for working with infinity);
- Solutions of several Diophantine equations, including instances of the Brahmagupta equation $Nx^2 + 1 = y^2$ (this is also known, wrongly, as the 'Pell equation'). For example, he solves the equation $61x^2 + 1 = y^2$, and finds its *least* positive integral solution: $x = 226153980,\ y = 1766319049$.

Isolated cases of higher degree equations occur (a cubic and a biquadratic, respectively), but the equations considered are not of a general nature.

Some Diophantine Problems from Bījagaṇita

A *Diophantine equation* or *Diophantine problem* is one for which we seek solutions in integers or in rational numbers; sometimes in only the positive integers. For example, we have the equation associated with the Pythagorean theorem (generally remembered as $a^2 + b^2 = c^2$), whose solutions yield the Pythagorean triples.

Among the many Diophantine problems posed and solved in *Bījagaṇita*, mention may be made of the following.

Problem 1: *Find four numbers whose sum is equal to the sum of their squares.* (This is Example 56 of *Bījagaṇita*.)

Bhāskarā offers the following solution. Take any four integers, say 1, 2, 3, 4. Their sum is 10, and the sum of their squares is 30. As $10 : 30 = 1 : 3$, we multiply all the numbers by the ratio $1 : 3$, and so arrive at the nice equality

$$\frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \frac{4}{3} = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{3}\right)^2 + \left(\frac{4}{3}\right)^2.$$

Simple and neat! Note that the method yields solutions in rational numbers, not integers.

This method clearly generalizes to any number of numbers. For example, one can solve problems of this kind: *Find two positive numbers for which the sum of their squares is equal to the sum of their cubes.*

Problem 2: *Find two numbers the sum of whose squares is a cube, and the sum of whose cubes is a square.* (This is Example 59 of *Bījagaṇita*.)

The way this is expressed in *Bījagaṇita* is: "The sum of the cubes of two numbers, and the sum of their squares is a cube. If you know such numbers, I shall consider you great among algebraists."

Let the numbers be x and y. Then $x^2 + y^2$ is a cube, and $x^3 + y^3$ is a square. To find such pairs of numbers, Bhāskarā makes use of the fact that 1 and 2 are two numbers the sum of whose cubes is a square (namely: $1^3 + 2^3 = 9 = 3^2$). So he puts $x = z^2$ and $y = 2z^2$. Then $x^3 + y^3 = z^6 + 8z^6 = 9z^6 = \left(3z^3\right)^2$ which is a square. Thus the second condition is automatically satisfied. This allows him to focus on just the first condition.

We now have: $x^2 + y^2 = z^4 + 4z^4 = 5z^4 = 5z \times z^3$. This is a cube if $5z$ is a cube, which it will be if z is of the form $25r^3$ for any integer r, giving $x = 625r^6$ and $y = 1250r^6$. So we can generate pairs of numbers with the required property. Thus, $r = 1$ yields the pair $\{625, 1250\}$, while $r = 2$ yields $\{40000, 80000\}$.

Remark. We can use other pairs of numbers (instead of $\{1, 2\}$) the sum of whose cubes is a square. For example: $\{11, 37\}$, with $11^3 + 37^3 = 228^2$, and $\{56, 65\}$, with $56^3 + 65^3 = 671^2$. The question can also be asked whether this problem has solution pairs which are not generated by this kind of logic.

Problem 3: *Find pairs of integers A, B such that $(A + B)^2 + (A + B)^3 = 2(A^3 + B^3)$.* (This is Example 89 of *Bījagaṇita*.)

In *Bījagaṇita* we read: "The square and cube of the sum of two numbers is equal to twice the sum of their cubes. O Mathematician, please state the numbers."

Bhāskarā takes the integers A and B to be given by $A = x + y$, $B = x - y$. Substituting these into the given equation, he gets $A + B = 2x$, $A - B = 2y$, and so:

$$A^3 + B^3 = 2(x^3 + 3xy^2).$$

Hence we have:

$$4x^2 + 8x^3 = 4x^3 + 12xy^2,$$
$$\therefore\ 4x + 4x^2 = 12y^2.$$

The quantity on the left is $(2x + 1)^2 - 1$. Let $X = 2x + 1$ and $Y = 2y$; then we get an instance of a Brahmagupta equation,

$$X^2 - 3Y^2 = 1,$$

and Bhāskarā is quite at home in this territory! Working backwards, he obtains solutions of the original equation.

(X, Y)	$(7, 4)$	$(97, 56)$	$(1351, 780)$	...
(x, y)	$(3, 2)$	$(48, 28)$	$(675, 390)$	...
(A, B)	$(5, 1)$	$(76, 20)$	$(1065, 285)$	...

The solution $(A, B) = (5, 1)$ corresponds to the equality

$$6^2 + 6^3 = 2(5^3 + 1^3),$$

while $(A, B) = (76, 20)$ corresponds to

$$96^2 + 96^3 = 2(76^3 + 20^3).$$

Problem 4: *Find all positive integers x such that $5x^4 - 100x^2$ is a square.* (This is Example 90 of *Bījagaṇita*.)

The original version: "Five times the fourth power of the unknown reduced by hundred times its square gives a square. O Mathematician, give that number quickly."

Let x be an integer with the stated property. Then $5x^2(x^2 - 20)$ is a square, hence $5(x^2 - 20)$ is a square. Let $5(x^2 - 20) = y^2$. Since y^2 is a multiple of 5, it must be that y itself is a multiple of 5. Let $y = 5u$. Then we have $x^2 - 20 = 5u^2$. From this we see that x too is a multiple of 5. Let $x = 5v$; then $5v^2 - 4 = u^2$. So we must solve the equation $u^2 - 5v^2 = -4$.

Yet again we get a Brahmagupta equation, which we solve by the usual means. Its solutions, and the corresponding values for x and y, are given below.

(v, u)	$(1, 1)$	$(4, 2)$	$(11, 5)$	$(29, 13)$	$(76, 34)$	$(199, 89)$	...
(y, x)	$(5, 5)$	$(20, 10)$	$(55, 25)$	$(145, 65)$	$(380, 170)$	$(995, 445)$	...

Hence, x-values that fit the original equation are: 5, 10, 25, 65, 170, 445, ...

Remark. If we list the u-values alone, we get the following list:

$$1,\ 2,\ 5,\ 13,\ 34,\ 89,\ 233,\ 610,\ \ldots.$$

These are alternate members of the Fibonacci sequence! Note the nice feature of the v-values too: 1, 4, 11, 76, 199, ..., with:

$$4 = 5 - 1,\ \ 11 = 13 - 2,\ \ 76 = 89 - 13,\ \ 199 = 233 - 34,\ \ \ldots.$$

So the v-numbers are the differences between successive pairs of alternate members of the Fibonacci sequence.

Problem 5: *Find all positive integers x for which both $3x + 1$ and $5x + 1$ are squares.* (This is Example 99 of *Bījagaṇita*.)

Let x be a positive integer such that $3x + 1 = u^2$, $5x + 1 = v^2$ where u, v are integers. Bhāskarā now supposes that u is of the form $3y + 1$. This yields

$$3x + 1 = (3y + 1)^2, \quad \therefore 3x = 9y^2 + 6y, \quad \therefore x = 3y^2 + 2y.$$

Substituting this into the relation $5x + 1 = v^2$ we get:

$$v^2 = 5\left(3y^2 + 2y\right) + 1 = 15y^2 + 10y + 1.$$

Multiplication by 15 on both sides yields:

$$15v^2 = 225y^2 + 150y + 15 = (15y + 5)^2 - 10.$$

Let $z = 15y + 5$. Then we have $z^2 = 15v^2 + 10$, and we get a Brahmagupta equation once again. Here are its solutions and the corresponding x-values:

(z, v)	$(5, 1)$	$(35, 9)$	$(275, 71)$	$(2165, 559)$	$(17045, 4401)$	...
x	0	16	1008	62496	3873760	...

If we list the z-values along we get the sequence

$$1,\ 9,\ 71,\ 559,\ 4401,\ \dots,$$

with a pleasing underlying pattern which allows us to guess the succeeding terms.

Remark. What happens if at the initial stage Bhāskarā supposes that u is of the form $3y+2$? Then he would get:

$$3x+1=(3y+2)^2, \quad \therefore 3x=9y^2+12y+3, \quad \therefore x=3y^2+4y+1.$$

Substitute this into the relation $5x+1=v^2$:

$$v^2=5\left(3y^2+4y+1\right)+1=15y^2+20y+6.$$

Multiplication by 15 on both sides yields:

$$15v^2=225y^2+300y+90=(15y+10)^2-10.$$

Let $z=15y+10$. Then we have $z^2=15v^2+10$, and we get the very same Brahmagupta equation as earlier! So they yield the same x-values.

We remark here that in general, Bhāskarā seems to be interested in listing an infinity of solutions, rather than a single isolated solution.

Bhāskarā II and the seeds of the Calculus

Most remarkably, Bhāskarā seems to have been practically on the doorstep of calculus — several centuries ahead of Newton and Leibnitz! The evidence for this claim is the following.

- He determines the surface area and volume of a sphere by 'calculus-like' methods which are nearly the same as those used by Archimedes. Thus, area is determined by dividing the surface into a large number of small parts, and likewise for volume.
- In constructing his table of sines, Bhāskarā states (as always, in verse form) that if x and x' are close to one another, then

  $$\sin x'-\sin x=(x'-x)\cos x.$$

  This is equivalent to the statement that "the derivative of the sine function is the cosine function", i.e.,

  $$\frac{d}{dx}(\sin x)=\cos x.$$

  The statement is found in his book on astronomy.
- Lastly, the following statements are found: "Where the motion of the planet is an extremum, there the fruit of its motion is absent" (that is, the motion is stationary), and: "At the commencement and end of retrograde motion, the apparent motion of the planet vanishes."

  These statements together with the examples given earlier strongly suggest the presence of the seeds of the differential and integral calculus in Bhāskarā's work.

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FOLDING A 45°, 60°, 75° TRIANGLE FROM A SQUARE SHEET OF PAPER IN 6 EASY STEPS !

1. Start with a square sheet

2. Fold it in half, make a crease, then unfold.

3. Fold a corner so that it falls on the crease.

4. Fold another corner to the same crease.

5. Fold the remaining portion as shown.

6. This triangle has the stated property!