Tale of a Quadrilateral and a Triangle

in the classroom

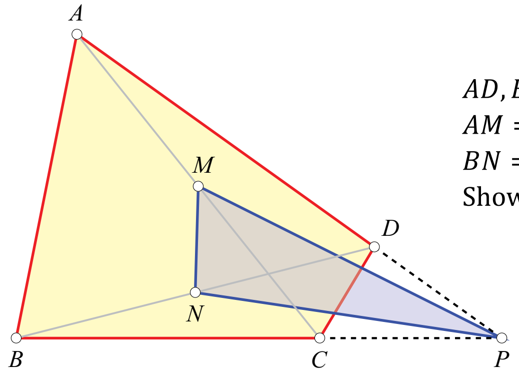
Bharat Karmarkar

This note is devoted to a proof of the following geometrical statement:

> *Let $ABCD$ be a convex quadrilateral in which AD is not parallel to BC. Let AD and BC meet, when extended, at P. Let M and N be the midpoints of diagonals AC and BD, respectively. Then the area of triangle PMN is one-quarter the area of quadrilateral $ABCD$.*

We present the proof in the form of pictures for which we give a light justification in each case. We use the following notation: if X denotes any plane geometric figure, then [X] denotes the area of X. So the square brackets stand for "area of …".

Proposition

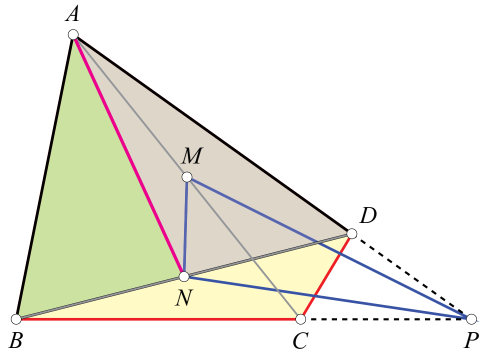


AD, BC meet at P
$AM = MC$
$BN = ND$
Show: $[PMN] = \frac{1}{4}[ABCD]$

Try to find your own proof before reading on!

Proof in Seven Movements

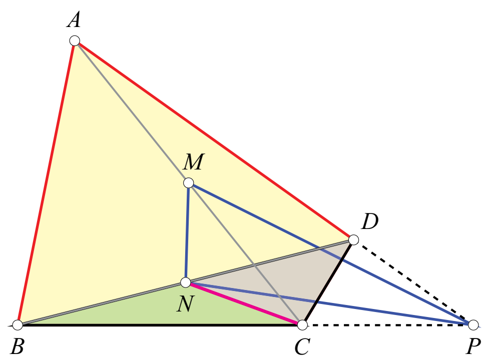
Step 1.



$[ABN] = [AND] = \frac{1}{2}\,[ABD]$.

Reason: AN is a median of $\triangle ABD$.

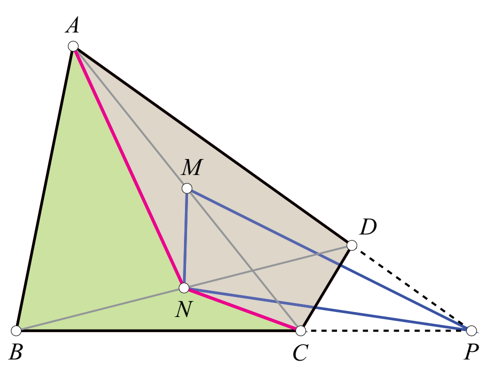
Step 2.



$[CBN] = [CND] = \frac{1}{2}\,[CBD]$.

Reason: CN is a median of $\triangle CBD$.

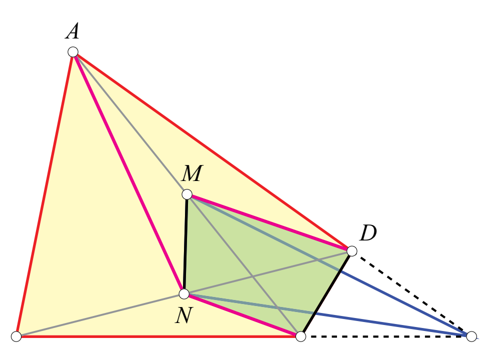
Step 3.



$[ANCD] = \frac{1}{2}\,[ABCD]$.

Proof: Follows by addition of the equalities in Steps 1 & 2.

Step 4.



$[CMN] = \frac{1}{2}\,[CAN]$

$[DMC] = \frac{1}{2}\,[DAC]$

Hence $[MNCD] = \frac{1}{2}\,[ANCD]$.

But $[ANCD] = \frac{1}{2}\,[ABCD]$.

Hence $[MNCD] = \frac{1}{4}\,[ABCD]$.

Step 5.

$$[PNB] = \tfrac{1}{2}[PDB],$$
$$[CNB] = \tfrac{1}{2}[CDB].$$

Now subtract:

$$[PNB] - [CNB] = [PNC],$$
$$[PDB] - [CDB] = [PDC].$$

Hence:

$$[PNC] = \tfrac{1}{2}[PDC].$$

Step 6.

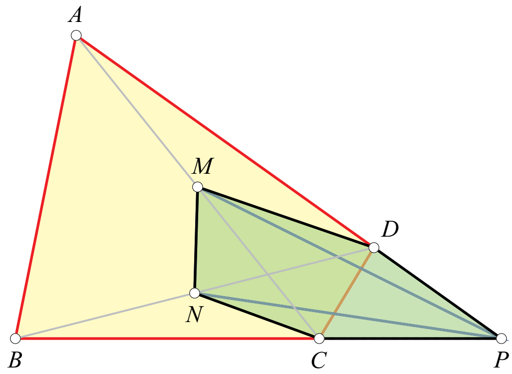
$$[PAM] = \tfrac{1}{2}[PAC],$$
$$[DAM] = \tfrac{1}{2}[DAC].$$

Now subtract:

$$[PAM] - [DAM] = [PDM],$$
$$[PAC] - [DAC] = [PDC].$$

Hence:

$$[PDM] = \tfrac{1}{2}[PDC].$$



Step 7.

Consider the polygon $PDMNC$. We have:

$$\begin{aligned}[PDMNC] &= [MNCD] + [PDC] \\ &= \tfrac{1}{4}[ABCD] + [PDC]. \qquad (1)\end{aligned}$$

We also have:

$$\begin{aligned}[PDMNC] &= [PMN] + [PDM] + [PNC] \\ &= [PMN] + \tfrac{1}{2}[PDC] + \tfrac{1}{2}[PDC] \\ &= [PMN] + [PDC]. \qquad (2)\end{aligned}$$

Comparing equalities (1) and (2), we get:

$$[PMN] = \tfrac{1}{4}[ABCD],$$

as required.

BHARAT KARMARKAR is a freelance educator. He believes that learning any subject is simply a tool to learn better learning habits and a better aptitude; what a learner really carries forward after schooling is *learning skills* rather than content knowledge. His learning club, located in Pune, is based on this vision. He may be contacted at learningclubpune@gmail.com.