in the classroom

Quadrilateral and Triangle: A Further Look

COMAC

The following proposition was proved in the preceding pages: *Let $ABCD$ be a convex quadrilateral in which AD is not parallel to BC. Let AD and BC meet, when extended, at P. Let M, N be the midpoints of diagonals AC, BD, respectively. Then* $[PMN] = \frac{1}{4}[ABCD]$. (Here square brackets denote area. See Figure 1.)

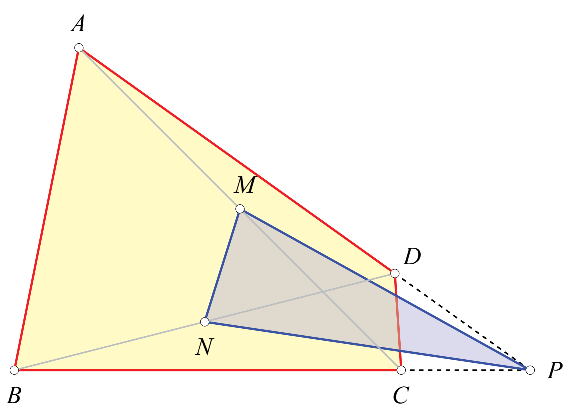

Figure 1.

It is of interest to look at this proposition through the lens given to us by George Pólya: that of tweaking a problem and seeing what we get. A very useful tweak is that of looking at *extreme situations*. In our context we identify the following extreme configurations when the quadrilateral $ABCD$ becomes 'degenerate' in some way:

(1) Quadrilateral $ABCD$ collapses into a triangle because two of its vertices coincide.
(2) Quadrilateral $ABCD$ collapses into a triangle because three of its vertices are collinear.

There are other possibilities, but we will mention them later.

Cases (1) and (2) can be considered as part of a continuum. We imagine that vertex D lies somewhere along segment AC. If D coincides with either A or C, we have case (1), and if D lies in the interior of segment AC, we have case (2).

The first possibility, of D coinciding with A, does not yield anything of interest, as line AD is undefined and hence point P is undefined as well. So we discard this.

If D coincides with C, we get a result which is well known; see Figure 2. For now, point P too coincides with C, which means that M is the midpoint of side AC and N is the midpoint of side BC. The statement that $[PMN] = \frac{1}{4}[ABCD]$ now simply reads: $[CMN] = \frac{1}{4}[CAB]$. This is easily seen to be true via the midpoint theorem.

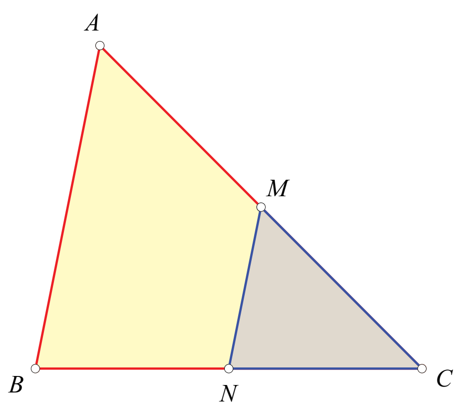

Figure 2.

In this figure, both D and P coincide with C. So M is the midpoint of AC, and N is the midpoint of BC. It is easy to see that $[CMN] = \frac{1}{4}[CAB]$.

It is always reassuring to find that a result being explored yields something well known as a special case. It means that the result under study cannot be completely wrong!

Of greater interest is the case when D lies in the interior of segment AC (Figure 3). Once again, P coincides with C. Constructing points M and N as earlier (M is the midpoint of AC and N is the midpoint of BD), the claim is: $[CMN] = \frac{1}{4}[CAB]$.

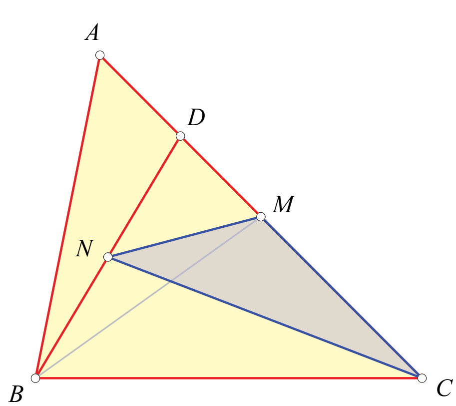

Figure 3.

In this figure, D is any point on AC; N is the midpoint of BD; M is the midpoint of AC. The claim is now: $[CMN] = \frac{1}{4}[CAB]$.

The claim is easy to prove:

$$
\begin{aligned}
[CMN] &= [CDN] - [MDN] \\
&= \frac{1}{2}[CDB] - \frac{1}{2}[MDB] \\
&= \frac{1}{2}[CMB] \;=\; \frac{1}{4}[CAB].
\end{aligned}
$$

Remark. There are two other ways in which the configuration under study can become special or degenerate:

(3) Quadrilateral $ABCD$ becomes a trapezium in which the sides AD and BC are parallel to each other (so they do not meet when extended).

(4) Quadrilateral $ABCD$ becomes a parallelogram.

But these cases are clearly rather troublesome. In case (3), the extended sides AD and BC fail to meet each other at all, so the point P does not exist. Or one may say that "P lies at an infinite distance along line BC (or line AD)". In case (4), the points M, N coincide; at the same time P lies at an infinite distance along line BC. (So (4) is in a way even "worse" than (3).)

A vector proof of the main proposition

We conclude with a vector proof of the proposition quoted at the start. Let position vectors of the various points in the diagram be with reference to P as the origin, and let the position vectors be denoted by lower case letters in boldface (Figure 4). Then:

$$
\begin{aligned}
2[PMN] &= \mathbf{m} \times \mathbf{n} \\
&= \frac{1}{2}(\mathbf{a}+\mathbf{c}) \times \frac{1}{2}(\mathbf{b}+\mathbf{d}), \\
\therefore\ 8[PMN] &= (\mathbf{a}+\mathbf{c}) \times (\mathbf{b}+\mathbf{d}) \\
&= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{d} + \mathbf{c}\times\mathbf{b} + \mathbf{c}\times\mathbf{d} \\
&= \mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{d}, \qquad \text{since } \{A, D, P\} \text{ and } \{B, C, P\} \text{ are collinear.} \\
\therefore\ 4[PMN] &= \frac{1}{2}(\mathbf{a}\times\mathbf{b}) - \frac{1}{2}(\mathbf{d}\times\mathbf{c}) \\
&= [PAB] - [PDC] \\
&= [ABCD].
\end{aligned}
$$

The smooth elegance of this proof is a testimony to the power of the vector approach.

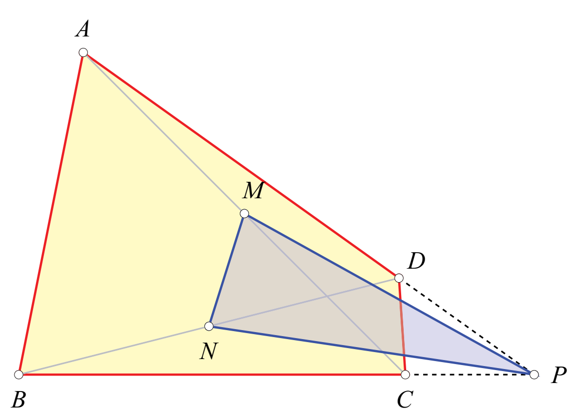

Figure 4.