Completing the Square…

A powerful technique, not a feared enemy!

in the classroom

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1. Introduction

The technique of completing the square is often introduced to high school students in the context of deriving the quadratic formula. Many students find the technique rather irritating, tedious and hard to remember and use appropriately. On the other hand, as a teacher I have grown to admire this technique for the range of problems it helps us solve and understand! I would like to share my perspective in the hope that it may save the idea of completing the square from being a dry technique to being a powerful tool with a rich history that solves many problems.

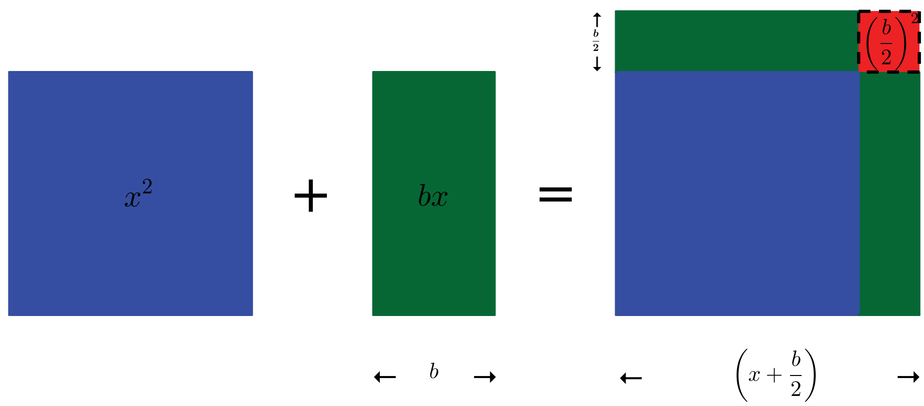
Completing the square helps us understand entirely the shapes of quadratic equations, locate the vertex of a parabola, derive the quadratic formula, find the range of a second-degree equation, find the inverse of a quadratic (with appropriate domain) and locate the centre and radius of a circle. What more can one ask for from a humble technique? Plenty more it seems! In the article "A Tale of Two Formulas" (*At Right Angles*, Vol. 2, No. 3, November 2013) many other applications are discussed.

This article repeats some of the ideas covered in our November 2013 issue, but since completing the square is such a rich idea, we have decided to explore it in greater detail in this issue.

2. The technique

Let us begin by looking at the technique. In its simplest form, the idea is to change an expression like $x^2 + bx + c$ into an expression of the form $(x - \alpha)^2 + \beta$. That is, a squared expression plus a pure number.

To do so we use a geometric illustration (Figure 1) similar to the one in the November 2013 issue, assuming for the moment $c = 0$. Here clearly $\alpha = -\frac{b}{2}$ and $\beta = \left(\frac{b}{2}\right)^2$. What is wonderful about this illustration is that it clearly explains why we often tell students: "to complete the square, add and subtract the square of half the coefficient of x"! Moreover, it now becomes clear why the technique is referred to as 'completing the square'.



$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

Figure 1.

If $c \neq 0$ then we get

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \frac{4c - b^2}{4} \tag{2.1}$$

Notice so far we have assumed that the coefficient of x^2 is 1. What if it is not? Perhaps as most of you would have guessed by now, that is not a big deal, we can always factor out the coefficient of x^2 and make it 1. So, if we have $ax^2 + bx + c$, we can rewrite it as $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ and all we need to do is replace the coefficient of x (that is, our old b) in equation 2.1 by $\frac{b}{a}$ and the constant term (our old c) by $\frac{c}{a}$ to get

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4\left(\frac{c}{a}\right) - \left(\frac{b}{a}\right)^2}{4}\right].$$

Simplifying we get

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \tag{2.2}$$

So once again we can write $ax^2 + bx + c$ as $a(x - \alpha)^2 + \beta$, where $\alpha = -\frac{b}{2}$ and $\beta = \frac{4ac - b^2}{4a}$.

We will now see how this amazing idea of representing a second degree equation in the above manner yields many applications.

3. All quadratics have the shape of a parabola

In this section we will show by the means of applying many transformations that the graphs of all quadratic equations of the form $y = ax^2 + bx + c$ have the same shape as that of $y = x^2$.

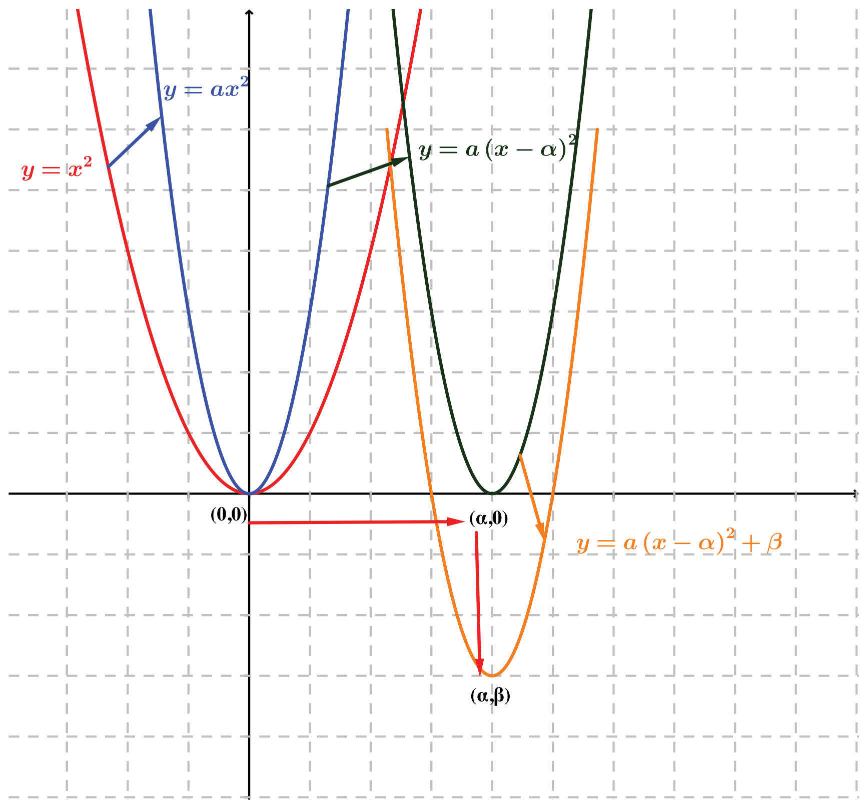


Figure 2.

The red graph (see Figure 2) is that of the function $y = x^2$, when we change the coefficient of x^2 from 1 to a, then the graph becomes narrower or broader depending on whether a is bigger than 1 or smaller than 1. Moreover, if a is positive then the graph is concave up (shaped like ∪) and if a is negative then the graph is concave down (shaped like ∩). The blue graph is that of $y = ax^2$. Notice the vertex of both the curves $y = x^2$ and $y = ax^2$ is $(0, 0)$. Now if we shift $y = ax^2$ to either the left or the right we get the green graph $y = a(x - \alpha)^2$. In this case the vertex has moved from $(0, 0)$ to $(\alpha, 0)$. If we now move the graph up and down then we get the orange graph whose equation is of the form $y = a(x - \alpha)^2 + \beta$. The vertex has now moved to (α, β). Readers familiar with computer software like GeoGebra can use sliders to change the values of a, α and β to see how the graphs move around. Can you explain why when a graph $y = f(x)$ is moved to the left or the right by α it becomes $y = f(x - \alpha)$, and when it is moved up or down it becomes $y = f(x) + \beta$?

Basically, what we have managed to do is to show that the graph of any function of the form $y = a(x - \alpha)^2 + \beta$ has the same shape as that of $y = x^2$. But in Section 2, using the technique of completing the square we saw that all quadratic expressions of the form $ax^2 + bx + c$ can be rewritten in the form $a(x - \alpha)^2 + \beta$! Which means that the graphs of all quadratic equations have the same shape as that of $y = x^2$, in other words that of a parabola.

What is very satisfying here is that we completely understand the graphs of all quadratic equations, without the use of calculus.

4. The quadratic formula for free!

Before we go on to other applications, let us quickly derive the quadratic formula! So if $ax^2 + bx + c = 0$ from equation 2.2 we have $a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a} = 0$. Rearranging we get $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$ and from there we get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

5. The vertex of a parabola and other applications

Let us come back to our equation $ax^2 + bx + c = a\left(x - \alpha\right)^2 + \beta$, where $\alpha = -\frac{b}{2}$ and $\beta = \frac{4ac-b^2}{4a}$. It is clear from Figure 2 that the vertex is (α, β). So if we are just given the function $f(x) = ax^2 + bx + c$, can we find the minimum or maximum value of $f(x)$? Obviously, before we have learned calculus!

Well! All we need to do is examine the expression $a\left(x - \alpha\right)^2 + \beta$. Let us assume first that $a > 0$. Then it is clear that no matter what value we plug in for x, since $a\left(x - \alpha\right)^2 > 0, f(x) > \beta$. Therefore the *minimum* value for $f(x)$ is β and this least value occurs when $x = \alpha$. Since we know that the shape of a quadratic is a parabola, the vertex then has to be (α, β). If we now assume $a < 0$, one can easily see that the *maximum* value of $f(x) = \beta$ and occurs once again when $x = \alpha$.

For example, consider the quadratic function $f(x) = -2x^2 - 4x + 1$. You can do the math and find that we can rewrite it as $f(x) = -2\left(x + 1\right)^2 + 3$. Therefore the vertex of the parabola is $(-1, 3)$ and the function has a maximum value of 3 at $x = -1$.

By completing the square it is now easy to see that the range of the above function is all real numbers less than or equal to 3. And if we are asked to find the inverse of $f(x) = -2x^2 - 4x + 1$, when $x \leq -1$ (making the function one-to-one), we can rearrange $y - 2\left(x - 1\right)^2 + 3$ to get $f^{-1}(x) = -1 - \sqrt{\frac{3-x}{2}}$. All this becomes very clear when we graph the functions concerned (see Figure 3).

One last application before we end this article with a historical note on completing the square. We all know that an equation in two variables like $x^2 + 2x + y^2 - 4y = 20$ is the equation of a circle. But, how do we find its centre and radius? Of course, by completing the square! I leave it to the reader to show that the circle given above has radius 5 and centre $(-1, 2)$.

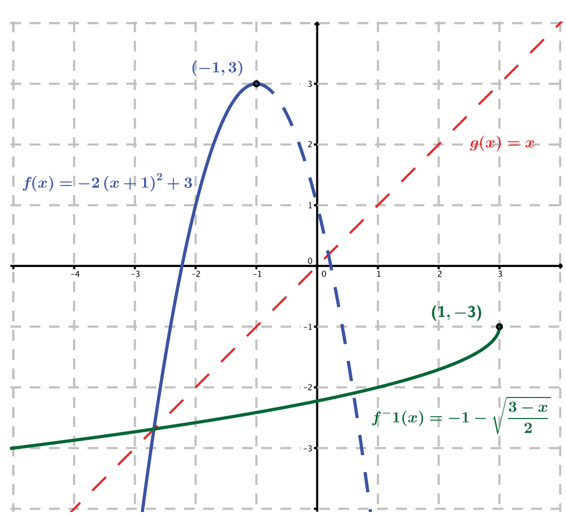


Figure 3.

6. Historical note

It appears that almost all ancient cultures, the Egyptian, Babylonian, Greek, Chinese and Indian, seem to have been aware of quadratic equations. Many of them obtained their solutions through the technique of completing the square. In this article I will touch upon some highlights that I found interesting.

We begin by describing Euclid's (300 BCE) method of completing the square (*The Elements*, Proposition 6, Book II). He solved the quadratic equation $x^2 + bx = c$, by completing the square as in Figure 4.

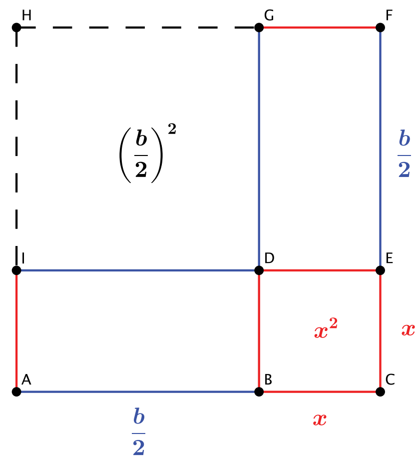


Figure 4.

The figure shows that $x^2 + bx + (\frac{b}{2})^2 = \left(x + \frac{b}{2}\right)^2$. Hence $c + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$, and from here we can find x. Exactly the same method can be found in a Mesopotamian tablet (BM13901, between 1800 BCE to 2000 BCE) with $b = \frac{3}{4}$ and $c = 1$. Euclid of course had generalized the construction and had provided a proof. Euclid also had an elaborate geometric construction for the solution of certain quadratic equations, but we will not get into them here.

We now move to the Indian contribution. Various books mention that quadratic equations show up in the *Sulba-sutras* and in the *Bakshali Manuscript*, but we will mention the work of Brahmagupta and Sridhara.

Brahmagupta (7th century CE) is credited with having derived the quadratic formula, perhaps using the technique of completing the square, and he may well have been the first mathematician to recognize that quadratic equations had two roots, and he permitted negative roots.

Sridhara (8th or 9th century CE) came up with the following ingenious trick, often referred to as 'Sridhara's rule'. This method can be used to derive the quadratic formula 'from scratch' without the use of fractions. His trick was to first multiply $ax^2 + bx + c = 0$ throughout by $4a$, yielding $4a^2x^2 + 4abx + 4ac = 0$, then adding b^2 to both sides, giving us $(2ax)^2 + 4abx + b^2 = b^2 - 4ac$. This in turn gives

$$(2ax + b)^2 = b^2 - 4ac$$

and from here the quadratic formula is a cinch! In fact, for the fun of it, why don't you illustrate this rule geometrically?

In the November 2013 issue al-Khwarizmi's (9th century CE) geometric solution of the equation $x^2 + 10x = 39$ was given, in fact he had solutions to six different types of quadratic equations (he did not permit negative solutions). I am sure many of you are aware of this, but it bears repeating, that the words 'Algebra' and 'Algorithm' come from al-Khwarizmi's name and his famous book *Hisab al-jabr w'al-muqabala*. Another Arab mathematician Thabit ibn Qurra (9th century CE) using Euclid's theorems showed how to solve the general quadratic equation geometrically.

The quadratic formula as it is currently used was published by the French mathematician Rene Descartes in *La Geometrie* in 1637.

We leave the readers with a wonderful problem from Bhaskaracharya's *Lilavati* (source: *The Crest of the Peacock* by George Gheverghese Joseph, Example 9.2, page 274) .

From a swarm of bees, a number equal to the square root of half the total number of bees flew out to the lotus flowers. Soon after $\frac{8}{9}$ *of the total swarm went to the same place. A male bee enticed by the fragrance of the lotus flew into it. But when it was inside the night fell, the lotus closed and the bee was caught inside. To its buzz, its consort responded anxiously from outside. O my beloved! How many bees are there?*

SHASHIDHAR JAGADEESHAN received his PhD from Syracuse University in 1994. He has been teaching mathematics for 25 years. He is a firm believer that mathematics is a human endeavour, and his interest lies in conveying the beauty of mathematics to students and demonstrating that it is possible to create learning environments where children enjoy learning mathematics. He is the author of *Math Alive!*, a resource book for teachers, and has written articles in education journals sharing his insights. He may be contacted at jshashidhar@gmail.com.