in the classroom

Solution to the 'Origamics' Problem

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In this note, we offer an explanation to the observations made in the 'Origamics' article (November 2013 issue of *At Right Angles*). The following observations had been made: (a) The points of intersection of the X-creases fall on the vertical midline of the square. (b) The points of intersection of the X-creases vary along a short distance from below the centre of the square. (c) The three lines connecting the point of intersection to the starting point on the edge and the two lower vertices are equal in length. Of these, (a) and (c) are easy to explain, while (b) offers greater challenge.

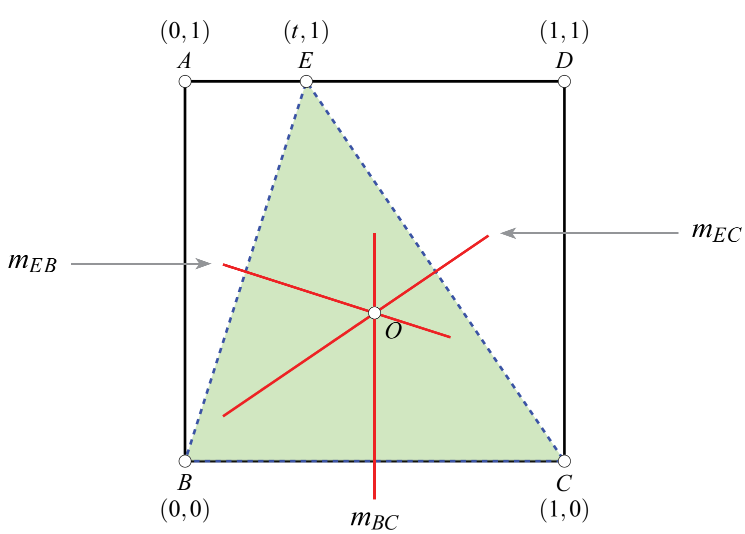

Figure 1.

In Figure 1, $ABCD$ is the square sheet, and E is the point on the upper side of the square which is folded successively to B and C. The fold lines (creases) are shown in red color. Segments EB and EC do not appear on the original sheet but we have drawn them (in dashed blue colour), this creating $\triangle EBC$. It should be clear that the X-creases are perpendicular bisectors of the sides EB and EC of $\triangle EBC$. These two lines necessarily meet on the perpendicular bisector of the third side BC of the triangle. This explains observations (a) and (c): why the X-lines meet on the midline of the square, and also why the point of concurrence of the lines is equidistant from E, B, C, for this point is the circumcentre of $\triangle EBC$.

To explain (b), we assign coordinates: $B = (0,0)$, $C = (1,0)$, $D = (1,1)$, $A = (0,1)$, $E = (t,1)$. Here $0 \leq t \leq 1$. The slope of EB is $\frac{1}{t}$, hence the slope of the perpendicular bisector of EB is $-t$. The coordinates of the midpoint of EB are $\left(\frac{t}{2}, \frac{1}{2}\right)$. Therefore the equation of the perpendicular bisector of EB (line m_{EB} in Figure 1) is

$$y - \frac{1}{2} = -t\left(x - \frac{t}{2}\right).$$

The equation of the perpendicular bisector of BC (line m_{BC} in Figure 1) is simply $x = \frac{1}{2}$. Let m_{EB} and m_{BC} meet at $O = (u, v)$; this point is, of course, the circumcentre of $\triangle EBC$. Since $u = \frac{1}{2}$, we have:

$$v = \frac{1}{2} - t\left(\frac{1}{2} - \frac{t}{2}\right) = \frac{1}{2} - \frac{t(1-t)}{2}.$$

Now $0 \leq t \leq 1$. The maximum and minimum values taken by $t(1-t)$ over $0 \leq t \leq 1$ are $\frac{1}{4}$ (taken when $t = \frac{1}{2}$) and 0 (taken when $t = 0$ and $t = 1$), respectively. To see why the maximum value is $\frac{1}{4}$, observe that

$$t(1-t) = t - t^2 = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2 \leq \frac{1}{4},$$

with equality just when $t = \dfrac{1}{2}$.

Hence v lies between $\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{4}$ and $\frac{1}{2}$, i.e.,

$$\frac{3}{8} \leq v \leq \frac{1}{2}.$$

Thus O travels on a line segment with endpoints $\left(\frac{1}{2}, \frac{3}{8}\right)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$. This short segment is the locus of O.

Remark. The above derivation shows that the expression $t(1-t)$ is symmetric about $t = \frac{1}{2}$. Another way of seeing this is to note that the replacement $t \mapsto 1-t$ leaves the expression $t(1-t)$ unchanged (the two factors simply swap places). The geometric expression of this symmetry is the following: if we reflect the entire configuration in the midline of the square (the line m_{BC}), triangle EBC is mapped to a triangle congruent to itself, so its circumcentre lies at the same height above BC as earlier.

Two more loci. It is of interest to inquire into two more loci: those traced out respectively by I, the incentre of $\triangle EBC$, and by H, the orthocentre

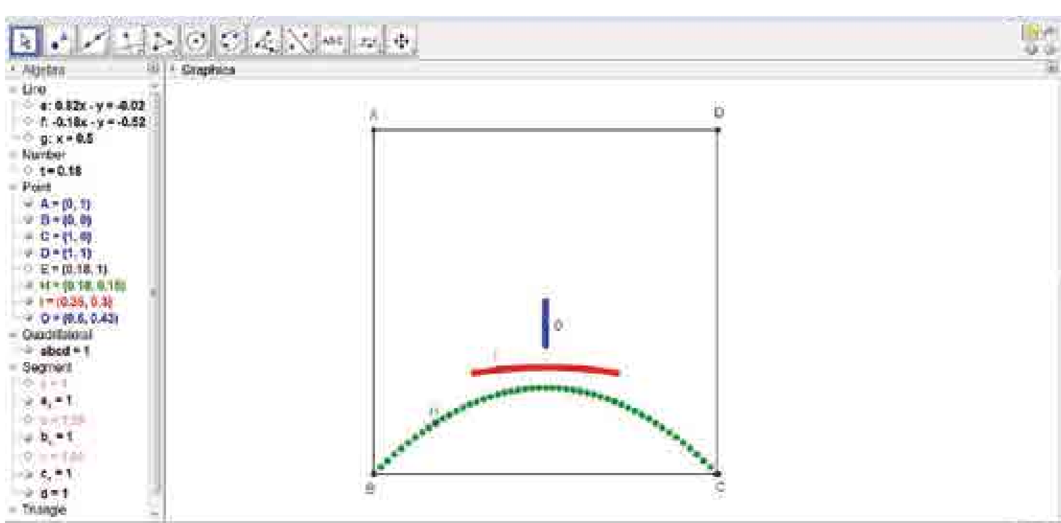

Figure 2. Three loci

of $\triangle EBC$. The latter locus is easily found. When $E = (t, 1)$, the equations of the altitudes through E and C are, respectively:

$$x = t, \quad y = -t(x - 1).$$

Solving these, we get $H = \left(t, t - t^2\right)$. So H travels on a parabolic arc with end-points B and C, with its highest point on the midline of the square, at height $\frac{1}{4}$.

The locus of I is a much less familiar curve. Using the vector formula for the coordinates of the incentre in terms of the coordinates of the vertices, we find that when $E = (t, 1)$, the coordinates of I are $(x(t), y(t))$, where

$$x(t) = \frac{t + \sqrt{1 + t^2}}{1 + \sqrt{1 + t^2} + \sqrt{1 + (1 - t)^2}},$$

$$y(t) = \frac{1}{1 + \sqrt{1 + t^2} + \sqrt{1 + (1 - t)^2}}.$$

The GeoGebra screen shot in Figure 2 shows all the three loci.

A suggested extension. This activity can be done using a rectangular sheet of paper as well. Try doing it with a rectangular sheet with point E on the shorter edge and then on the longer edge. Also try it with a long thin rectangle, e.g., an A4 sheet halved along its shorter edge. Check whether you get configurations with the same features as described above.

An interesting question: Find the dimensions which the rectangle must have for the point O to lie outside the paper. (In this case the folds will not meet on the sheet at all.) Modify the algebraic calculations (done above) for the case of a rectangle with sides a and b, and check whether the calculation gives back the above formulas for the case $a = b = 1$.

A SLICE OF HISTORY: THE BIRTH OF LOGARITHMS

The following extract is taken from the 'Futility Closet' (http://www.futilitycloset.com/about/) and its page http://www.futilitycloset.com/2014/09/11/likewise/?utm_source=rss&utm_medium=rss&utm_campaign=likewise.

A charming little scene from mathematical history — in 1615 Gresham College geometry professor Henry Briggs rode the 300 miles from London to Edinburgh to meet John Napier, the discoverer of logarithms. A contemporary witnessed their meeting: of logarithms. A contemporary witnessed their meeting :

He brings Mr. Briggs up into My Lord's chamber, where almost one quarter of an hour was spent, each beholding the other with admiration, before one word was spoke: at last Mr. Briggs began. 'My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto Astronomy, viz. the Logarithms: but my Lord, being by you found out, I wonder nobody else found it before, when now being known it appears so easy.'

Their friendship was fast but short-lived: The first tables were published in 1614, and Napier died in 1617, perhaps due to overwork. In his last writings he notes that "owing to our bodily weakness we leave the actual computation of the new canon to others skilled in this kind of work, more particularly to that very learned scholar, my dear friend, Henry Briggs, public Professor of Geometry in London."

To find out more about the early history of logarithms, please refer to any of the following:

- http://www.maa.org/publications/periodicals/convergence/logarithms-the-early-history-of-a-familiar-function-introduction
- Shailesh Shirali, A Primer on Logarithms (Universities Press)
- http://www.westcler.org/gh/outtda/pdf_files/History_of_Logarithms.pdf