in the classroom

How To Prove It

This continues the 'Proof' column begun earlier. In this 'episode' we study some results from geometry related to the theme of concurrence.

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Concurrence of lines. An extremely common theme in plane geometry is that of proving the concurrence of three or more lines. (The dual problem: proving the collinearity of three or more points.) It is of interest to study the different strategies used. We study some well known results in this area and contrast the approaches used to prove them.

Perpendicular bisectors

Perhaps the easiest of all results on concurrence is this: ***The perpendicular bisectors of the three sides of a triangle concur.***

This is best proved using the idea of a locus, namely: ***Given two distinct points* A *and* B*, the locus of points* P *such that* PA = PB *is the perpendicular bisector of segment* AB**; see Figure 1 (a). Here's how the proof uses this locus idea.

Let the perpendicular bisectors of sides AB and AC meet at O; see Figure 1 (b). Join OA, OB, OC. Then $OA = OB$, and also $OA = OC$, hence $OB = OC$. The last equality means that O is equidistant from B and C and hence lies on the perpendicular bisector of BC. Therefore the three perpendicular bisectors meet in a point.

How can we be sure that the perpendicular bisectors of sides AB and AC do meet? That's easy: they meet because they are not parallel to each other, and this is ensured by the fact that they are respectively perpendicular to AB and AC; and AB and AC are

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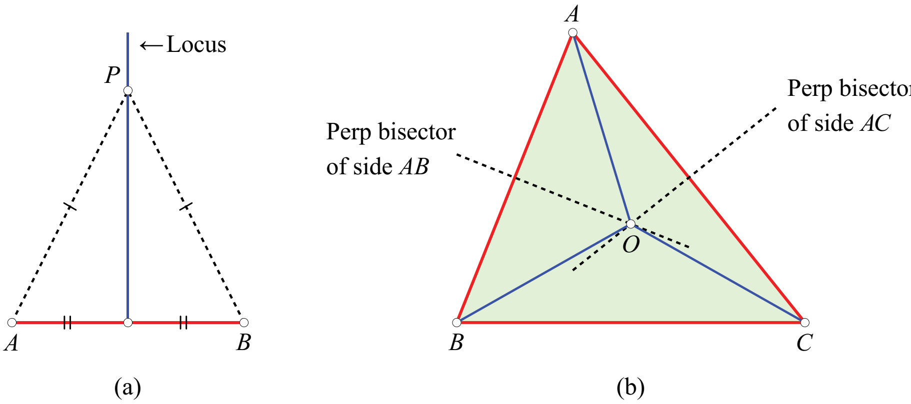

Figure 1. Concurrence of the perpendicular bisectors of the sides of a triangle

certainly not parallel to each other — after all, they meet at A!

Internal angle bisectors

Of the same nature, and proved the same way, is this: ***The internal bisectors of the three angles of a triangle concur.***

To show this, we use a different locus fact which just as basic: ***Given an angle* ABC*, the locus of points* P *equidistant from the arms* BA *and* BC *is the internal bisector of* ∡ABC.** See Figure 2 (a).

Let the internal bisectors of $\measuredangle ABC$ and $\measuredangle ACB$ meet at I. Draw perpendiculars ID, IE, IF from I to sides BC, CA, AB respectively; see Figure 2 (b). Using the locus fact stated above, we see that $ID = IF$ and $ID = IE$. Hence $IE = IF$, which implies that I lies on the internal bisector of $\measuredangle BAC$. So the three internal angle bisectors meet in a point.

How can we be sure that the internal bisectors of $\measuredangle ABC$ and $\measuredangle ACB$ do meet? The lines must meet because they are not parallel to each other, and we can be sure of this because $\measuredangle ABC + \measuredangle ACB < 180°$, which implies that $\measuredangle IBC + \measuredangle ICB < 90°$.

A small tweak in this argument yields a related but slightly less familiar result: ***Given any triangle, the external bisectors of any two of its angles and the internal bisector of the third angle concur***; see Figure 3.

A general remark. The above two proofs have a common theme; namely, to prove that two quantities u and v are equal, show that both of them are equal to a third quantity w. Viewed thus in generality, we see a theme used frequently in mathematics, at all levels.

Medians

The result generally encountered next is:

Theorem. The medians of a triangle concur.

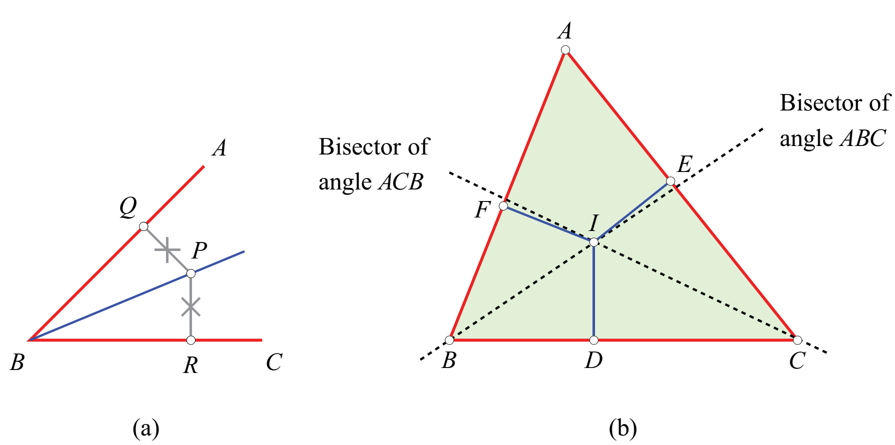

Figure 2. Concurrence of the internal bisectors of the angles of a triangle

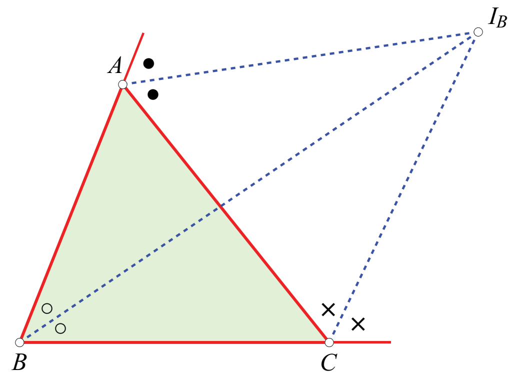

Figure 3. Concurrence of two external angle bisectors and one internal bisector

This turns out to be more challenging to prove, because the median is not so obviously a locus, unlike the perpendicular bisector of a line segment or the bisector of an angle. We offer three proofs of concurrence. The first is based on a well known theorem of elementary geometry: ***The segment joining the midpoints of two sides of a triangle is parallel to and half the third side.*** This is, of course, the **midpoint theorem**.

Proof based on the midpoint theorem. In Figure 4 (a) we have drawn two medians, BE and CF; they intersect at G_1. Consider $\triangle G_1EF$ and $\triangle G_1BC$. Since $EF \parallel BC$ (midpoint theorem), the two triangles are similar to each other, hence their sides are in proportion. Since $EF = \frac{1}{2}BC$ (midpoint theorem, again), it follows that $G_1E = \frac{1}{2}G_1B$. Hence G_1 divides segment BE in the ratio $2 : 1$.

In Figure 4 (b) we have drawn BE and the remaining median AD; they intersect at G_2. Considering $\triangle G_2AB$ and $\triangle G_2DE$ and invoking the midpoint theorem twice, like earlier, we deduce that G_2 divides segment BE in the ratio $2 : 1$.

So G_1 and G_2 divide BE in the identical ratio $(2 : 1)$. This means that they are the same point! In other words, the point where BE and CF meet is identical to the point where BE and AD meet. This obviously means that AD, BE, CF concur.

Remark. Note the reasoning involved. It is rather more subtle than the reasoning used in the proofs for concurrence of the perpendicular bisectors and the angle bisectors. The underlying principle by which we concluded that points G_1, G_2 are the same is illustrated in Figure 5. Let AB be a given segment, and let P be a variable point located strictly in its interior. (This means that P cannot coincide with either A or B.) Consider the ratio $t = AP/PB$. *Then, as P moves from A towards B, this ratio assumes every possible positive value exactly once.* No

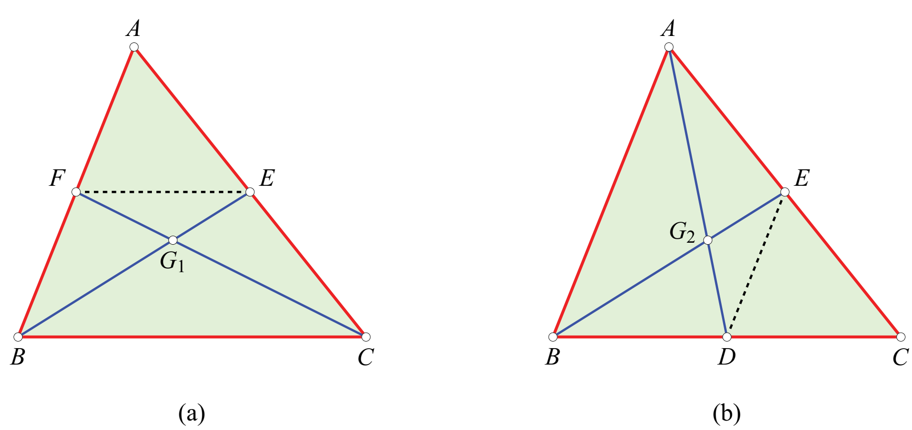

Figure 4. Concurrence of the medians of a triangle: proof using similar triangles

Figure 5. Variation of the ratio $t = AP/PB$ as P moves along AB

ratio is repeated. If P is very close to A, the ratio is close to 0, and it can get as close to 0 as we may want. Similarly, if P is very close to B, the ratio becomes very large, and it can get as large as we may want (there is no upper bound).

Or, to use terminology generally heard in a calculus class rather than a geometry class, we may say that the ratio $t = AP/PB$ is a strictly *monotonic function* of the position of P.

Proof based on area considerations. We offer a second proof which is of a very different nature. It uses one basic idea over and over again: *A median of a triangle divides it into two triangles with equal area*. (See Figure 6. Observe the notation used carefully: if X is any geometric figure, then $[X]$ denotes the area of X.)

Here's how we invoke this idea. In Figure 7 we have shown $\triangle ABC$ with two medians BE and CF which intersect at G. The line through A and G is then drawn; it intersects BC at D. Note that there is nothing being said about AD being a median and (therefore) D being the midpoint of BC. Rather, we have to *prove* that AD is a median. To make sure we do not fall into the trap of assuming implicitly that it is a median (and thus assuming the very thing we wish to prove), we have drawn it using a dashed line.

The various lines drawn within $\triangle ABC$ create six smaller triangles. Certain pairs of these are immediately seen to have equal area, by making use of the property mentioned above. (i) Since BE is a median, and G is a point on BE, $[GCE] = [GAE] = x$, say. (ii) Since CF is a median, and G is a point on CF, $[GAF] = [GBF] = y$, say. In Figure 7 we have written these symbols within the respective regions. Let $u = [GBD]$ and $v = [GCD]$, respectively. Since $[BCE] = [BAE]$, we have $u + v + x = y + y + x$; hence:

$$u + v = 2y.$$

Again, since $[CAF] = [CBF]$, we have $u + v + y = x + x + y$; hence:

$$u + v = 2x.$$

From the above equalities we deduce that

$$x = y.$$

Now we argue algebraically. Let $BD : DC = t$. Since $\triangle GBD$ and $\triangle GCD$ have equal altitude, and their bases are in the ratio $t : 1$, their areas bear this same ratio to one another. Hence:

$$u = tv.$$

Similarly, the areas of $\triangle ABD$ and $\triangle ACD$ bear this same ratio to one another. Hence:

$$2y + u = t(2x + v).$$

By subtraction the above two equalities yield:

$$2y = 2tx, \qquad \therefore\ y = tx.$$

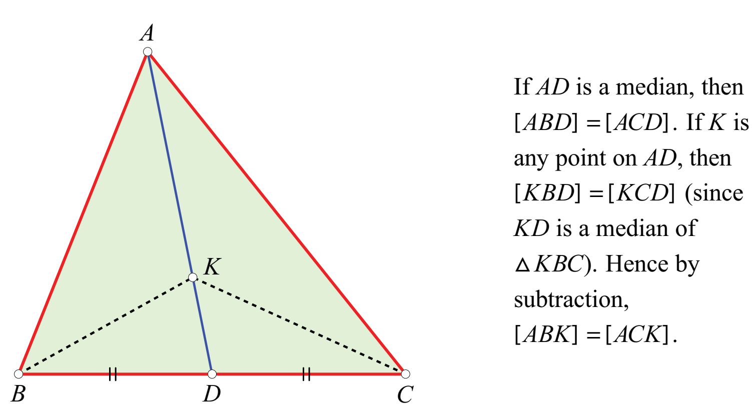

Figure 6. Property of a median of a triangle

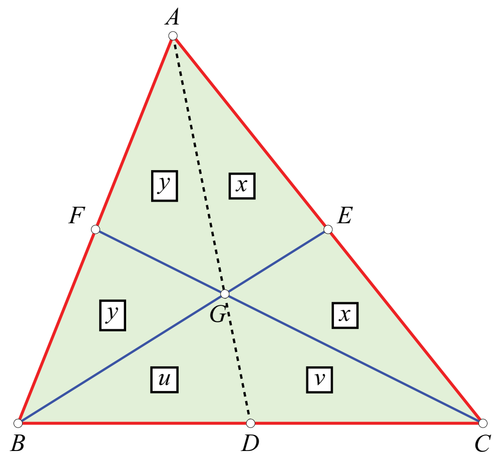

Figure 7. Making use of the area bisection property to prove concurrence

But we have already shown that $x = y$. Hence $t = 1$. But this means that D is the midpoint of BC. Hence AD is a median of $\triangle ABC$.

It follows that the third median passes through the point of intersection of BE and CF. That is, the medians of the triangle concur.

Remark. Note the use of algebraic manipulations in this proof. As such, this is not a "pure geometry" proof, and purists will frown at it. But the central idea is, surely, a pleasing one. We hope you enjoyed this neat interplay of algebra and geometry.

Third proof. We now offer a third proof which draws upon the midpoint theorem for its central logic (just like the first proof shown above) but in a very different way. It is taken from an article that appeared in *The Mathematical Gazette*, in the November 2001 issue, written by Mowaffaq Hajja and Peter Walker, and titled "Why Must the Triangle's Medians Be Concurrent?" The central idea is ingenious and subtle.

Figure 8(a) shows a triangle ABC and points D, E, F which are the midpoints of its sides BC, CA, AB; the median AD has been drawn, and the medial triangle DEF shown shaded. Here's how we argue.

- ⋆ The midpoint theorem implies that figure $DEAF$ is a parallelogram. Since the diagonals of a parallelogram bisect each other, we deduce that AD bisects side EF of $\triangle DEF$. So the medial line through A of $\triangle ABC$ is the same line as the medial line through D of $\triangle DEF$.

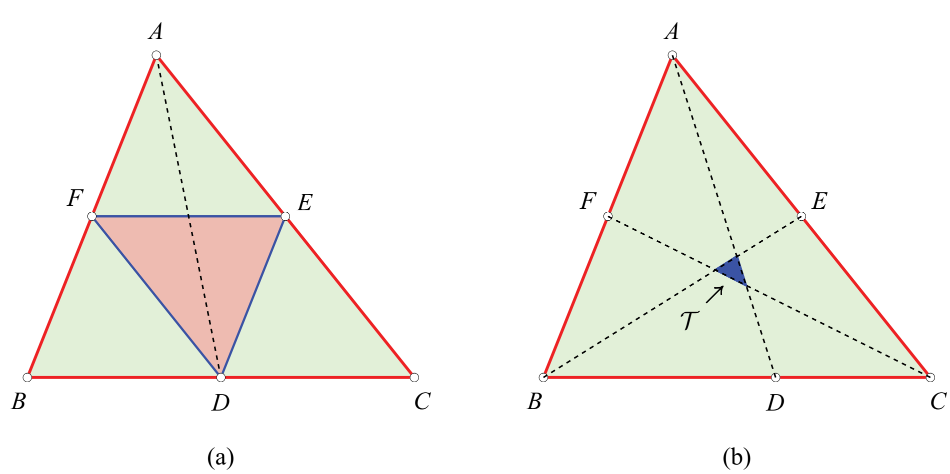

Figure 8.

- It follows, by symmetry, that: *the medial lines of $\triangle DEF$ coincide with the medial lines of $\triangle ABC$*. We now make use of this observation in a surprising way.
- Next, we note that $\triangle DEF$ is similar to $\triangle ABC$: it is a copy of $\triangle ABC$ but with half its scale. It stands to reason, surely, that any genuine geometric property exhibited by $\triangle ABC$ will also be exhibited by $\triangle DEF$. In particular we can be sure that if the medians of $\triangle ABC$ concur, then so do the medians of $\triangle DEF$; and if the medians of $\triangle ABC$ do not concur, then neither do the medians of $\triangle DEF$.
- Let us now suppose now that the medians of $\triangle ABC$ do *not* meet in a point, and rather that the situation is as depicted in Figure 8 (b). In that case, it must be that the medians AD, BE, CF enclose a triangle $\mathcal{T}$ (shown with a heavy blue filling).
- Since $\triangle DEF$ is similar to $\triangle ABC$, we expect that the triangle $\mathcal{T}'$ created by the medial lines of $\triangle DEF$ is similar to $\mathcal{T}$ but has half its dimensions. This implies that $[\mathcal{T}'] = \frac{1}{4}[\mathcal{T}]$.
- On the other hand, we just noted that the medial lines of $\triangle DEF$ coincide with those of $\triangle ABC$. This means that $\mathcal{T}'$ coincides with $\mathcal{T}$. Therefore we have: $[\mathcal{T}'] = [\mathcal{T}]$.
- How can we reconcile these two statements? There is clearly just one way: it must be that both $[\mathcal{T}] = 0$ and $[\mathcal{T}'] = 0$. In other words, both $\mathcal{T}$ and $\mathcal{T}'$ have zero area.
- But this is just another way of saying that the medians AD, BE, CF concur!

Closing remark. One proposition, but three entirely different proofs .... How would you contrast them?

In a future column, we will discuss an important tool in the study of concurrence, which allows us to deduce a vast number of concurrence results in one stroke.

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