Angle Bisectors in a Quadrilateral

A Ramachandran

The bisectors of the interior angles of a quadrilateral are either all concurrent or meet pairwise at 4, 5 or 6 points, in any case forming a cyclic quadrilateral. The situation of exactly three bisectors being concurrent is not possible. See Figure 1 for a possible situation. The reader is invited to prove these as well as observations regarding some of the special cases mentioned below.

Start with the last observation. Assume that three angle bisectors in a quadrilateral are concurrent. Join the point of

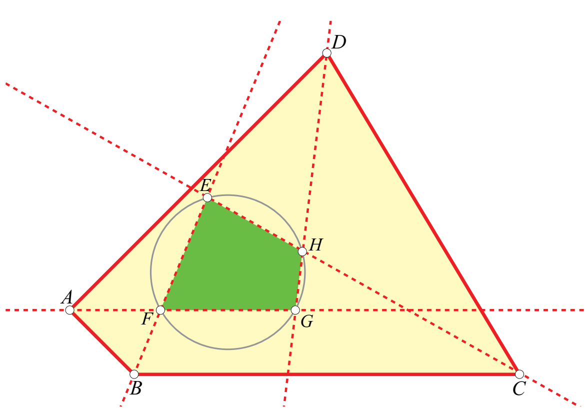


Figure 1. A typical configuration, showing how a cyclic quadrilateral $EFGH$ is formed

Keywords: *Quadrilateral, diagonal, angular bisector, tangential quadrilateral, kite, rhombus, square, isosceles trapezium, non-isosceles trapezium, cyclic, incircle*

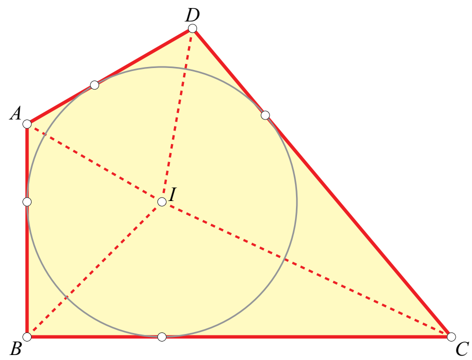


Figure 2. A tangential quadrilateral

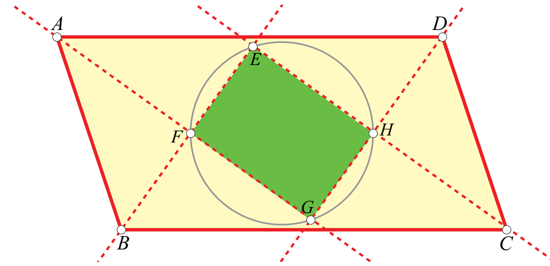


Figure 3. If $ABCD$ is a parallelogram, then $EFGH$ is a rectangle

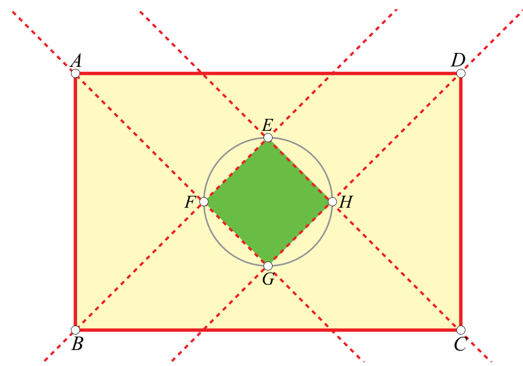


Figure 4. If $ABCD$ is a rectangle, then $EFGH$ is a square

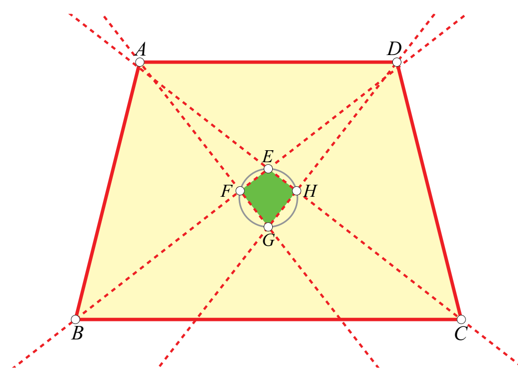


Figure 5. If $ABCD$ is an isosceles trapezium with opposite side sums unequal, then $EFGH$ is a cyclic kite

concurrence to the fourth vertex. Prove that this line indeed bisects the angle at the fourth vertex.

A quadrilateral in which all the four angle bisectors meet at a point is a *Tangential quadrilateral* — one which has an circle touching all the four sides. This circle is the *incircle* of the quadrilateral, and its centre is the *incentre* of the quadrilateral (see Figure 2). Prove that in such a quadrilateral the sums of the lengths of opposite sides are equal (i.e., in Figure 2, $AB + CD = AD + BC$). Also prove the converse: If the sums of the lengths of opposite sides of a quadrilateral are equal, the quadrilateral is tangential.

Special cases of tangential quadrilaterals are the kite and rhombus (including the square). In the former, one diagonal bisects the angles at the vertices it joins, while this is true of both diagonals in a rhombus.

In the case of the bisectors meeting pairwise they form a quadrilateral. Prove that this is cyclic. Also prove the following:

(1) The angle bisectors in a general parallelogram form a rectangle (see Figure 3).

(2) The angle bisectors in a general rectangle form a square (see Figure 4).

(3) The angle bisectors in an isosceles trapezium (opposite side sums unequal) form a cyclic kite (also called a 'right kite'), where the equal angles are right angles (see Figure 5).

The case of a non-isosceles trapezium is shown in Figure 6. Here we get a quadrilateral in which one pair of opposite angles are right angles.

(4) In the case of a quadrilateral with only one set of opposite angles equal (opposite side sums unequal), the angle bisectors form an isosceles trapezium (see Figure 7).

Remark. We see from this survey that the simple act of drawing the internal bisectors of the four angles of a quadrilateral produces a configuration that offers unexpected riches. We invite the reader to supply proofs for the many claims made.

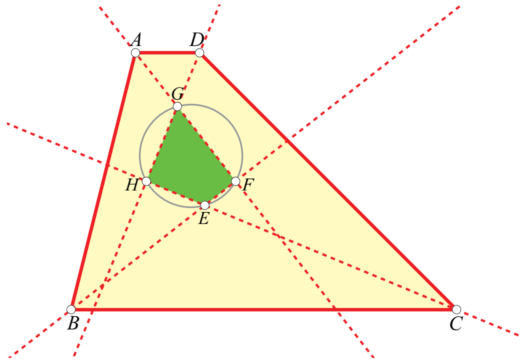


Figure 6. The case when $ABCD$ is a non-isosceles trapezium: the result is that $EFGH$ is a cyclic quadrilateral in which $\angle F = \angle H = 90°$

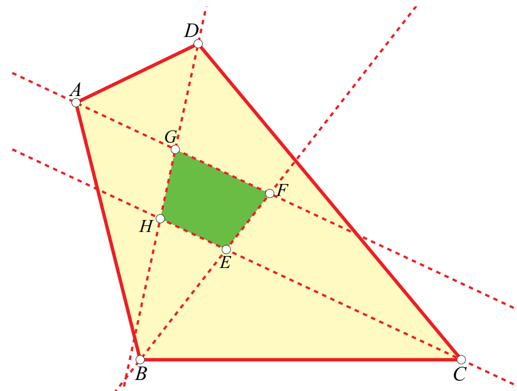
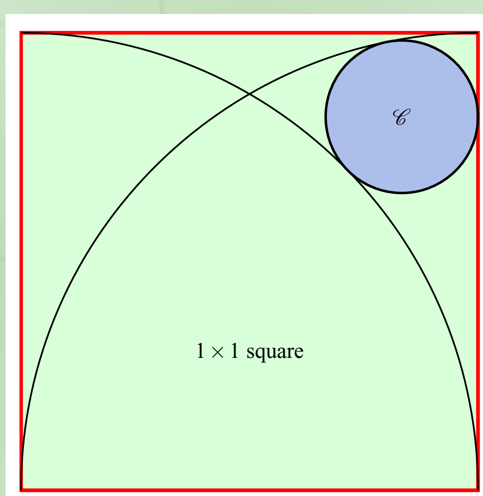


Figure 7. The case when $ABCD$ has $\angle B = \angle D$ but $\angle A \neq \angle C$: the result is that $EFGH$ is an isosceles trapezium ($FG \parallel EH$ and $EF = HG$)

A RAMACHANDRAN has had a long standing interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for over two decades, and now stays in Chennai. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

CHALLENGE



Can you find the radius of the inscribed circle $\mathscr{C}$?
The two quarter circles
(centred at the two bottom vertices of the square) have radius 1 unit each.