feature

The Man

The Methods of Archimedes

The Message

Amrutha Manjunath

Introduction

Elsewhere in this issue is a review of *The Sand Reckoner* by Gillian Bradshaw. That review and this article are dedicated to one of the most celebrated mathematicians in the world. Archimedes is perhaps most famous for the discovery of the Archimedes Principle and the invention of levers, pulleys, pumps, military innovations (like the siege engines) and the Archimedean Screw. His mathematical contributions include approximations of π and $\sqrt{3}$ accurate to several decimal places, proof of the quadrature of the parabola, formula for the area of a circle, and formulae of surface areas and volumes of several solid shapes. In this article, I have focused on two techniques (called Archimedes' Methods) by which he arrived at the formula of the volume of a sphere.

In October 1998, a French family in New York put a thousand-year-old manuscript up for public auction. This manuscript, which the family had acquired in the 1920s, turned out to be a lost Archimedean palimpsest. Byzantine monks in the 13th century had washed the original mathematical text and reused the parchment for Christian liturgical writings. In the early 20th century, Johan Heiberg had studied the same manuscript at Constantinople (present-day Istanbul) and

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identified it for the first time as work by Archimedes. It disappeared for several years during the aftermath of the Greco-Turkish War, and resurfaced in the possession of the French businessman whose descendants put it up for auction. From 1999 to 2008, the manuscript was subject to extensive imaging study and conservation at the Walters Art Museum in Baltimore in collaboration with scientists at Rochester Institute of Technology and Stanford University. Many Archimedean texts were recovered from this palimpsest, of which the work on Methods is especially interesting to many mathematicians.

The Method of Exhaustion

The Method of Exhaustion is a well-known technique using which the area of a figure can be found by visualizing it to be composed of constituent polygons that converge to the area of the containing shape. It is considered to be the ancient-Greek equivalent of the modern notion of limits. Among other results, Archimedes used the Method of Exhaustion to compute the volume of a sphere. I have discussed this method below using modern notation.

Consider the hemisphere in Figure 1. Archimedes imagined the hemisphere to be formed by the layering of cylinders inscribed within the solid. Let the radius of the hemisphere be r, and radii of each cylinder be $r_1, r_2, r_3, \ldots, r_n$. If there are n cylinders of equal height laid one on top of one another, it follows that the height of each cylinder is r/n. By the Pythagorean Theorem:

$$r_1 = r^2 - \frac{r^2}{n^2}, \quad r_2 = r^2 - \frac{(2r)^2}{n^2}, \quad \ldots,$$
$$r_{n-1} = r^2 - \frac{(n-1)r^2}{n^2}, \quad r_n = r^2 - \frac{(nr)^2}{n^2}.$$

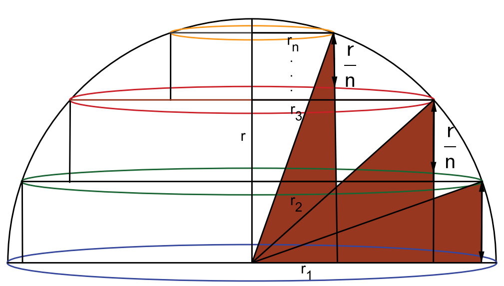

Figure 1.

As the number of cylinders increases, and the height of each cylinder correspondingly decreases, the sum of volumes of the cylinders is a closer and closer approximation to the volume of the hemisphere. Therefore, as n approaches infinity, the sum of the volumes of the cylinders equals the volume V of the hemisphere. That is,

$$\begin{aligned} V &= \lim_{n\to\infty}\left(\pi r_1^2 \frac{r}{n} + \pi r_2^2 \frac{r}{n} + \pi r_3^2 \frac{r}{n} + \cdots + \pi r_n^2 \frac{r}{n}\right) \\ &= \lim_{n\to\infty} \pi \frac{r}{n}(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2). \end{aligned}$$

Substitute $r_j = r^2 - (jr/n)^2$ for $1 \le j \le n$:

$$\begin{aligned} V &= \lim_{n\to\infty} \pi \frac{r^3}{n}\left(n - \frac{(1^2 + 2^2 + 3^2 + \cdots + n^2)}{n^2}\right) \\ &= \pi r^3 - \pi r^3 \lim_{n\to\infty} \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right). \end{aligned}$$

Now use the formula
$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$:

$$\begin{aligned} V &= \pi r^3 - \pi r^3 \lim_{n\to\infty}\left(\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\right) \\ &= \pi r^3 - \pi r^3 \lim_{n\to\infty}\left(\frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right) \\ &= \pi r^3 - \frac{\pi r^3}{3} = \frac{2\pi r^3}{3}. \end{aligned}$$

Since V is half the required volume, the volume of the sphere with radius r is given by $\frac{4}{3}\pi r^3$.

A similar argument can be made to obtain the same result if the hemisphere is thought to be circumscribed by a layering of cylinders; see Figure 2. The solid shape is in fact "sandwiched" between the inscribed and circumscribed cylinders. As n tends to infinity, the two stacks of cylinders converge to the form of the hemisphere.

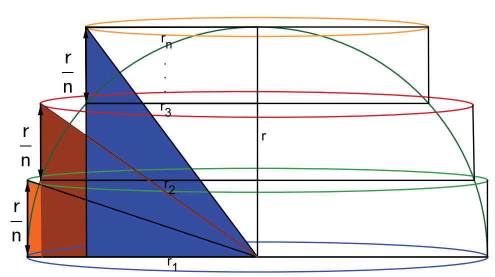

Figure 2.

The Method of Equilibrium

Now I will discuss the second technique by which Archimedes arrived at the same result for the volume of a sphere. This technique, known as the Method of Equilibrium, was found in the lost palimpsest. It sheds light on a very uniquely Archimedean way of thinking about surface areas and volumes of solid shapes, and employs an argument that resonates with the modern notion of integral calculus.

There has been considerable debate among mathematicians about which Method of Archimedes is the superior one. Archimedes conceptualized notions of limits and integration well before calculus emerged as a powerful mathematical tool, so both methods contain ideas much ahead of their time. Historians of mathematics such as Howard Eves argue that the Method of Exhaustion is "sterile" because its elegance is apparent only if the result is already known. While this is debatable, the Method of Equilibrium is unique for Archimedes' use of mechanics to prove a purely mathematical result. Archimedes himself is said to have preferred the Method of Exhaustion because he felt that it was mathematically more rigorous. Perhaps this was born out of his innate preference for pure mathematics to mechanical inventions. However, in the words of E.T. Bell, "To a modern all is fair in love, war, and mathematics." Maybe the equilibrium argument is considered more elegant today because there is something enchanting when borders between related disciplines melt to reveal how closely interlinked the disciplines really are.

To find the volume of a sphere by the Method of Equilibrium, it helps to think of the solid as cut up into a large number of very thin strips hung end to end on an imaginary lever. This proof compares the moments of two solids when placed on the lever. Since volume is proportional to mass, moment of the solid can be defined as the product of its volume and lever length (the distance from the point about which the shapes are hung to the centroid of the volume).

Figure 3 is a cross-sectional view along the equator of the sphere. Here $AO = OB = 2r$. Consider the cylinder and cone of revolution obtained by rotating rectangle $OPSB$ and triangle OCB about the AB axis. Suppose thin vertical slices of thickness Δx are cut from the three solids at distance x from O. The approximate volumes of the sections of each solid are deduced to be:

Sphere: The equation of the circular cross-section of the sphere is $(x-r)^2 + y^2 = r^2$, i.e., $y^2 = x(2r - x)$. Therefore the volume of revolution of the slice of sphere with thickness Δx and height y is $\pi y^2 \Delta x = \pi x(2r - x)\Delta x$.

Cone: The volume of revolution of the slice of cone with thickness Δx and height x is $\pi x^2 \Delta x$.

Cylinder: The volume of revolution of the slice of cylinder with thickness Δx and height $2r$ is $\pi r^2 \Delta x$.

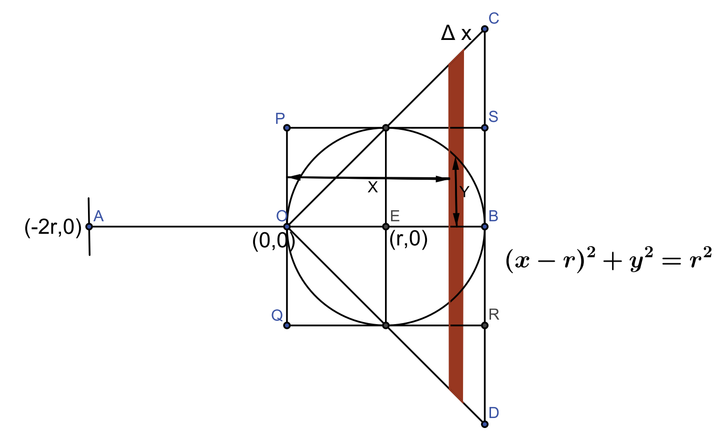

Figure 3.

If the slices from the sphere and the cone are imagined to be stacked at A, they form a single point mass. Their combined moment about the point O is given by:

$$\begin{aligned}&\text{Sum of volumes of slices of sphere and cone}\\&\quad\times \text{length } OA = (\pi x(2r-x)\Delta x + \pi x^2\Delta x)2r\\&\quad = 4\pi r^2 x\Delta x.\end{aligned}$$

The moment about O of the slice cut from the cylinder (when its position is unchanged) is given by:

$$\begin{aligned}&\text{Volume of slice of cylinder}\\&\quad\times \text{distance from } O \text{ to slice} = (\pi r^2\Delta x) * (x)\\&\quad = \pi r^2 x\Delta x.\end{aligned}$$

Therefore, the moment about O of the slices of the cone and sphere is 4 times the moment about O of the slice of the cylinder. When a large number of such slices are added together, the following expression is obtained:

$$\begin{aligned}&2r \times \text{volume of sphere} + \text{volume of cone}\\&\quad = 4r \times \text{volume of cylinder.}\end{aligned}$$

Here, $4r$ is the length of the lever arm. Since the volume of the cone is known to be $\frac{\pi(2r)^3}{3}$ and that of the cylinder to be $(2r)\pi r^2 = 2\pi r^3$, we get:

$$2r \times \text{volume of sphere} + \frac{8\pi r^3}{3} = 8\pi r^4.$$

Therefore, the volume of the sphere is $\frac{4\pi r^3}{3}$.

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feature

3, 4, 5 ... And other memorable triples — Part II

SHAILESH SHIRALI

In Part I of this article we had showcased the triple $(3, 4, 5)$ by highlighting some of its properties and some configurations where it occurred naturally. We now attempt to extend this to other triples of consecutive integers. To begin with, we study the two 'siblings' of $(3, 4, 5)$, namely, the triples $(2, 3, 4)$ and $(4, 5, 6)$. We start first with the elder sibling, $(4, 5, 6)$. (We do need to show the older ones some respect, don't we?)

The triple 4, 5, 6

In Figure 1 we see a sketch of a triangle ABC with sides $4, 5, 6$ (with $a = 6, b = 5, c = 4$). Is there anything special about the triangle? Let's do some exploration using *GeoGebra*.

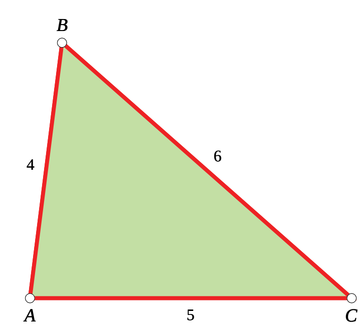


Figure 1.

Keywords: *Triangle, consecutive integers, triple, double angle, sine rule, cosine rule, Pythagoras*

Figure 1 shows a *GeoGebra* sketch of the triangle. We start by measuring the angles of the triangle (using the tool available in *GeoGebra*). Here is the output:

$$\measuredangle A = 82.82°, \qquad \measuredangle B = 55.77°, \qquad \measuredangle C = 41.41°.$$

Examining the data, we quickly notice that 82.82 is twice 41.41, in other words: $\measuredangle A = 2\measuredangle C$. Right away we have uncovered something notable and of interest!

But wait: this relation has been *numerically determined*. Could it be the case that if we compute both angle measures to more decimal places than shown above, the above relation will turn out to be only approximate and not exact? How can we check whether or not $\measuredangle A$ is *exactly* twice $\measuredangle C$?

We can do so using trigonometry. Let us compute the cosines of all three angles of the triangle using the cosine rule:

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{25+16-36}{2\times 20} = \frac{1}{8},$$
$$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{16+36-25}{2\times 24} = \frac{9}{16},$$
$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{36+25-16}{2\times 30} = \frac{3}{4}.$$

To see if $\measuredangle A = 2\measuredangle C$ as suggested by the empirical evidence, we must check whether $\cos A = 2\cos^2 C - 1$ (for we have the identity $\cos 2\theta = 2\cos^2\theta - 1$ which is true for any angle θ). We have:

$$2\cos^2 C - 1 = 2\left(\frac{3}{4}\right)^2 - 1 = \frac{9}{8} - 1 = \frac{1}{8} = \cos A,$$

and since both $\measuredangle A$ and $\measuredangle C$ are acute angles, the verification is complete. *So the relation $\measuredangle A = 2\measuredangle C$ is indeed exact.*

The same property can be proved by a geometric argument which may be preferred by some. In Figure 2 (a) we have redrawn the 4-5-6 triangle with the perpendicular BD from vertex B to side AC. Our first task is to find the length x of AD. We shall make use of the Pythagorean theorem to do so. Let h be the length of BD. Then we have:

$$h^2 + x^2 = 4^2,$$
$$h^2 + (5-x)^2 = 6^2,$$

hence by subtraction: $(5-x)^2 - x^2 = 6^2 - 4^2$, i.e., $25 - 10x = 20$. This yields $x = 1/2$.

Let E be the point on side AC such that $AE = 1$ unit; see Figure 2 (b). Join BE. Since $DE = DA$, it follows that $BE = BA$. Also $EC = 5 - 1 = 4$ units. So we have $AB = BE = EC$. Hence $\measuredangle BEA = 2\measuredangle BCA$, and also $\measuredangle BEA = \measuredangle BAE$. It follows that $\measuredangle BAC = 2\measuredangle BCA$, i.e., $\measuredangle A = 2\measuredangle C$.

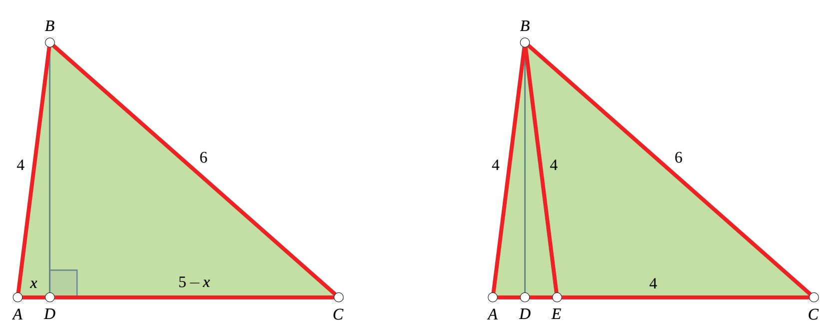
A Stronger Property

We now prove something much more striking:

Theorem 1. *There is only one triple of consecutive integers with the property that the triangle with these numbers as its side lengths has one angle which is twice another one. This is the triple* $(4, 5, 6)$.

Let the sides of the triangle be $n, n+1, n+2$. Let the triangle be labelled ABC so that $a = n+2$, $b = n+1$, $c = n$. Since $a > b > c$, we have $\measuredangle A > \measuredangle B > \measuredangle C$. So if one angle of the triangle is twice another, one of the following must be true: (i) $\measuredangle A = 2\measuredangle B$ (ii) $\measuredangle B = 2\measuredangle C$ (iii) $\measuredangle A = 2\measuredangle C$.

There are now two ways of proceeding. One is to use the cosine rule. This works, but the algebra is messy. The other, which is more interesting as well as more efficient, and which we prefer, is to use a geometric Pythagoras-style theorem which is striking by itself.



(a) (b)

Figure 2.

Theorem 2. *Let $\triangle ABC$ have sides a, b, c. Then the relation $\measuredangle A = 2\measuredangle B$ is true if and only if* $a^2 = b(b+c)$.

Proof of Theorem 2: Forward implication. We first tackle the statement: if $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$. (This is the 'only if' part of the theorem.) We offer a trigonometric proof of the result. Let $\measuredangle B = \theta$; then $\measuredangle A = 2\theta$ and $\measuredangle C = 180° - 3\theta$. Hence we have $\sin A = \sin 2\theta$ and $\sin C = \sin 3\theta$. The sine rule yields:

$$\frac{a}{\sin 2\theta} = \frac{b}{\sin\theta} = \frac{c}{\sin 3\theta}.$$

From the first equality we get:

$$a = b \cdot \frac{\sin 2\theta}{\sin\theta} = 2b\cos\theta, \qquad \therefore\ \cos\theta = \frac{a}{2b}.$$

The second equality yields:

$$\begin{aligned} c = b \cdot \frac{\sin 3\theta}{\sin\theta} &= b \cdot \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta} \\ &= b\left(3 - 4\sin^2\theta\right) \\ &= b\left(4\cos^2\theta - 1\right). \end{aligned}$$

Substituting for $\cos\theta$ in this relation, we get:

$$c = b\left(\frac{a^2}{b^2} - 1\right) = \frac{a^2-b^2}{b},$$
$$\therefore\ a^2 = b^2 + bc = b(b+c),$$

as claimed.

Proof of Theorem 2: Reverse implication. Now we tackle the 'if' part of the theorem, namely: if $a^2 = b(b+c)$, then $\measuredangle A = 2\measuredangle B$. Once again, we offer a trigonometric proof of the result. We use the sine rule together with the following beautiful identity whose proof we leave as an exercise:

$$\sin^2 A - \sin^2 B = \sin(A+B)\ \sin(A-B).$$

The sine rule tells us that for any triangle ABC, we have $a/\sin A = b/\sin B = c/\sin C =$ some constant k. (In fact, k is the circum-diameter of the triangle, i.e., it is twice the radius of the circumcircle. But we do not need this information right now.)

From the relation $a^2 = b(b+c)$ we get $a^2 - b^2 = bc$, which tells us that $a > b$ and therefore that $\measuredangle A > \measuredangle B$. The same relation also yields, by the sine rule:

$$\sin^2 A - \sin^2 B = \sin B\ \sin C.$$

Using the trigonometric identity quoted above, we get:

$$\sin(A+B)\ \sin(A-B) = \sin B\ \sin C.$$

Since $A + B + C = 180°$, we have $\sin(A+B) = \sin C$. Since $\sin C \neq 0$, we get:

$$\sin(A-B) = \sin B.$$

Since $A - B$ and B lie between 0° and 180° and have equal sine, they are either equal angles or they are supplementary angles. The latter possibility leads to $(A-B) + B = 180°$, i.e., $A = 180°$, which is absurd. Hence this case does not hold. It follows that $A - B = B$, i.e., $\measuredangle A = 2\measuredangle B$.

There is also an elegant geometric proof of the result (both parts: forward implication as well as reverse implication), which we shall discuss later.

Proof of Theorem 1. We now use Theorem 2 to prove Theorem 1. We consider the three possibilities in turn.

Case (i): If $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$, hence:

$$(n+2)^2 = (n+1)(2n+1),$$
$$\therefore\ n^2 + 4n + 4 = 2n^2 + 3n + 1,$$
$$\therefore\ n^2 - n - 3 = 0.$$

This equation has roots $n = \frac{1}{2}(1 \pm \sqrt{13})$. These are not positive integers (or even rational numbers), so we do not get any solution from this possibility.

Case (ii): If $\measuredangle B = 2\measuredangle C$, then $b^2 = c(c+a)$, hence:

$$(n+1)^2 = n(2n+2),$$
$$\therefore\ (n-1)(n+1) = 0.$$

This yields $n = \pm 1$. Only the positive sign is of interest to us. However, the triangle corresponding to $n = 1$ has sides 1, 2, 3 and so is degenerate: it is 'flat', with angles 180°, 0° and 0°. Note that the solution is not 'wrong'. For, this triangle has $\measuredangle B = 0° = \measuredangle C$, which means that we do have the relation $\measuredangle B = 2\measuredangle C$! But it is of no interest to us, so we move on.

Case (iii): If $\measuredangle A = 2\measuredangle C$, then $a^2 = c(c+b)$, hence:

$$\begin{aligned}(n+2)^2 &= n(2n+1),\\ \therefore\ n^2+4n+4 &= 2n^2+n,\\ \therefore\ n^2-3n-4 &= 0,\\ \therefore\ (n+1)(n-4) &= 0.\end{aligned}$$

The last equation has roots $n = -1$ and $n = 4$. We finally do get a positive integral root, $n = 4$, and this yields a genuine, well-behaved triangle: a triangle with sides $4, 5, 6$. This yields a solution to the stated problem.

It follows that there is precisely one triangle with the stated property: the one that has sides $4, 5, 6$.

In closing we may say that the triple $(4, 5, 6)$ can lay its own claim to fame, with its own pleasing property, just like its better known sibling $(3, 4, 5)$.

A Geometric Proof of Theorem 2

Some readers may prefer to see a *geometric* proof of Theorem 2 (we had earlier given a proof using trigonometry). We offer one such proof here.

First we deal with the forward implication: if $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$. The relevant configuration is shown in Figure 3.

We need an auxiliary construction. Draw a circle tangent to side BC at B and passing through vertex A. (The circle may be constructed as follows: draw a perpendicular to BC through B, and draw the perpendicular bisector of side AB; the point where these two lines meet is then the centre of the desired circle. These auxiliary construction lines have not been shown in Figure 3, to avoid a visual clutter.)

Extend side CA beyond vertex A to meet the circle again at point D. Draw segments BD and AD, as shown. Let AD have length d. Let $\measuredangle ABC = \theta$; then $\measuredangle BAC = 2\theta$ as per the given data.

From the fact that CB is tangent to the circle at B, two deductions follow: (i) $\measuredangle ABC = \measuredangle ADB$, i.e., $\measuredangle ADB = \theta$; this follows from the "angle in the alternate segment" theorem; (ii) $CB^2 = CA \times CD$, i.e., $a^2 = b(b+d)$; this is true because CAD is a secant.

Since $\measuredangle BAC = \measuredangle ADB + \measuredangle ABD$, it follows that $\measuredangle ADB = \theta$. Hence $\triangle ADB$ is isosceles, with $AD = AB$. So $d = c$. Combining this with deduction (ii), above, we see that $a^2 = b(b+c)$, as claimed.

Now for the reverse implication: if $a^2 = b(b+c)$, then $\measuredangle A = 2\measuredangle B$. We use the same figure for the proof, with the same auxiliary construction. The configuration is depicted in Figure 4. As earlier, we have drawn a circle tangent to side BC at B and passing through vertex A; then we have extended side CA beyond vertex A to meet the circle again at point D, and drawn segments BD and AD. Let AD have length d.

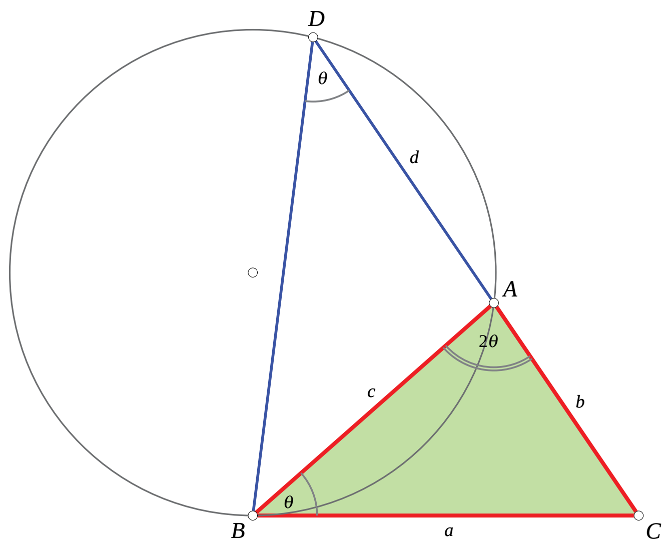


Figure 3. Given that $\measuredangle A = 2\measuredangle B$, to show that $a^2 = b(b+c)$

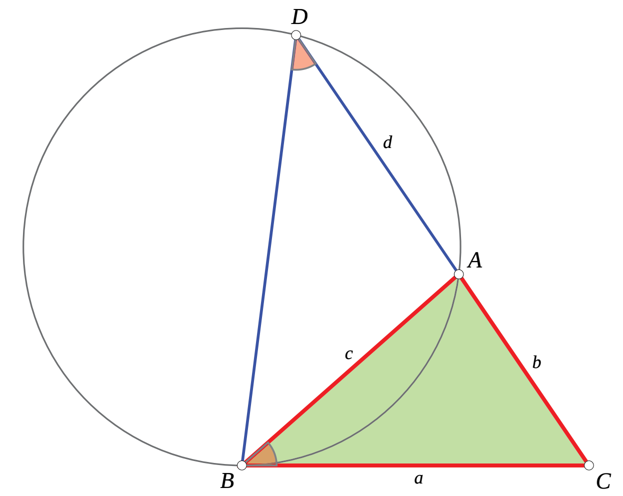


Figure 4. Given that $a^2 = b(b+c)$, to show that $\measuredangle A = 2\measuredangle B$

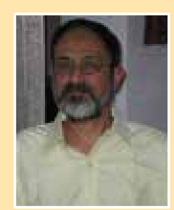
Since CB is tangent to the circle at B, and CAD is a secant, we have the following relation: $CB^2 = CA \times CD$, i.e., $a^2 = b(b+d)$.

But we also have the given relation $a^2 = b(b+c)$. Comparing the two relations, we conclude that $c = d$, i.e., $AB = AD$. Hence $\measuredangle ABD = \measuredangle ADB$. And since $\measuredangle BAC = \measuredangle ABD + \measuredangle ADB$, it follows that $\measuredangle BAC = 2\measuredangle ADB$.

But we also have $\measuredangle ABC = \measuredangle ADB$, by the "angle in the alternate segment" theorem. Hence $\measuredangle BAC = 2\measuredangle ABC$, i.e., $\measuredangle A = 2\measuredangle B$, as claimed.

Appendix: Integer triples associated with this theorem

Associated with the Pythagorean theorem we have the number theoretic problem of generating Pythagorean triples. In the same way, associated with the main result derived in this article, we have another interesting number theoretic problem: that of generating integer triples (a, b, c) which satisfy the equation $a^2 = b(b+c)$. We may want to impose the additional condition that a, b, c are coprime, just as we did in the case of Pythagorean triples. We already have one example of such a triple: $(6, 4, 5)$. Are there any others? Yes; and they are quite easy to find. We leave this question for the reader to tackle: that of finding an efficient and effective algorithm for generating all coprime integer triples (a, b, c) which satisfy the equation $a^2 = b(b+c)$. We will take up a study of this equation in a subsequent article.



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