feature

Lurking within any triangle . . .

Morley's Miracle - Part I

. . . is an equilateral triangle

In this three-part series we study one of the most celebrated and beautiful theorems of Euclidean geometry, discovered at the dawn of the twentieth century. It has justly become known as 'Morley's Miracle'. It happens to be uncommonly challenging to prove!

The inaugural July 2012 issue of this magazine had displayed a figure of the theorem on the cover, and we had promised to present a proof in a later issue.
It is appropriate that we are making good this promise now.

V G Tikekar

Frank Morley (1860–1937) was British by birth (though he lived much of his life in USA) and an expert geometer. After getting a Sc.D. degree from Cambridge, he became Chairman of the Math Dept of the Johns Hopkins University in 1900. Prior to this, while working as a Professor of Mathematics at Haverford College (Pennsylvania), he had found a theorem in the field of Euclidean geometry (but using methods far removed from those of synthetic geometry). It was regarded by many as "marvelous but strange" because it related to the lines that trisect the internal angles of a triangle. Early in the nineteenth century it had been shown that it is not possible to trisect a general angle using straightedge and compass (thus settling a question that had been lying unresolved since Greek times), leading to a sort of psychological barrier in exploring any matter related to angle trisection. Morley's theorem thus came as a shock to many lovers of mathematics. Over time the result became known as 'Morley's Miracle'.

Keywords: *Angle trisectors, equilateral, cardioid*

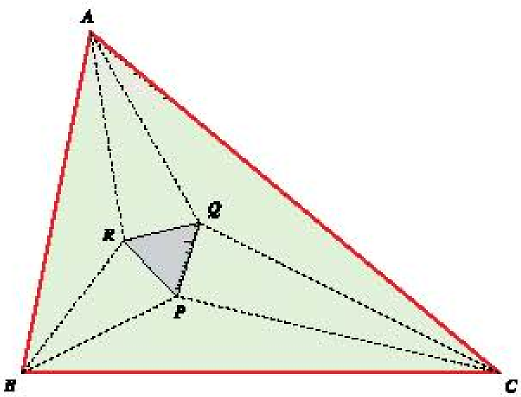

FIGURE 1. The angle trisectors closest to each side intersect in points which are the vertices of an equilateral triangle

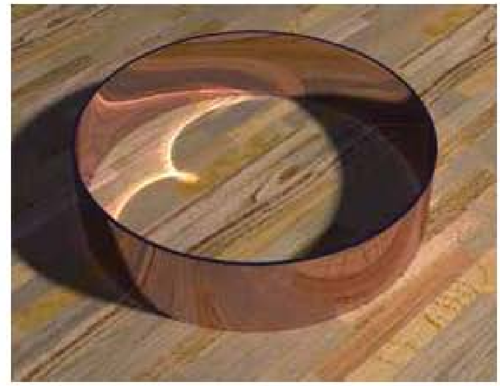

FIGURE 2. A manifestation of the coffee-cup caustic curve. Source: http://m.today.duke.edu/2009/04/caustics.html

Displayed below is the statement of the theorem.

Theorem 1 (Morley's Miracle). *In any triangle, the three points of intersection of the adjacent angle trisectors closest to each side are the vertices of an equilateral triangle.*

The theorem is illustrated in Figure 1: the trisectors of $\angle A$ are lines AQ and AR; those of $\angle B$ are BP and BR; and those of $\angle C$ are CP and CQ. The trisectors nearest to side BC meet at P, those nearest to side CA meet at Q, and those nearest to side AB meet at R. The claim now is ΔPQR is always equilateral, regardless of the shape of ΔABC. Readers are encouraged to make trial drawings of their own (this is very conveniently done using *GeoGebra*) to see that the claim does seem to be true.

The theorem is indeed a gem; but it is also a great challenge to prove. In this three-part series we describe a few of the many known proofs.

It appears that Morley discovered (and proved) the result around the year 1899, during his investigations on the differential geometry of curves, but did not publish it anywhere. He did mention it to a few people, though, and it gradually became known. His proof did not use the methods we associate with Euclidean geometry; it was based on a study of the set of cardioids that touch all the three sides of a triangle! (A cardioid is a heart-shaped curve generated by each point on a circle that rolls without slipping on a fixed circle of equal radius. You will see a portion of a cardioid on the surface of a cup of milk or coffee when light shines upon it; see Figure 2. So it is sometimes called the 'coffee cup curve'.)

The first 'pure geometry' proof to be published was in 1909, by M.T. Naraniengar (who, incidentally, was president of the Indian Mathematical Society from 1930 to 1932, and Editor of the Journal of the Indian Mathematical Society from 1909 to 1927). In 1914 one more such proof appeared, by Marr and Taylor. Over the years a large number of beautiful proofs have been found, of which special mention must be made of one by John Conway (see [4]).

Naraniengar's proof This beautiful proof follows an unusual strategy, one not seen too often in geometric proofs.

The trisectors closest to side BC meet at P (as stated). Let the remaining two trisectors of $\angle B$ and $\angle C$ (respectively) meet at S. Join SP (see Figure 3). In ΔSBC, rays BP and CP are internal angle bisectors, so P is the incentre of ΔSBC; therefore, SP bisects $\angle BSC$.

Now, locate points R' on BS and Q' on CS such that $\angle SPR' = 30°$ and $\angle SPQ' = 30°$. Then $\Delta SPR' \cong \Delta SPQ'$ (angle-side-angle congruence), so $PR' = PQ'$. Hence $\Delta PQ'R'$ is an isosceles triangle with a 60° angle; but this implies that it is equilateral. Hence to prove Morley's theorem, it suffices to show that AR' and AQ' are trisectors of $\angle A$.

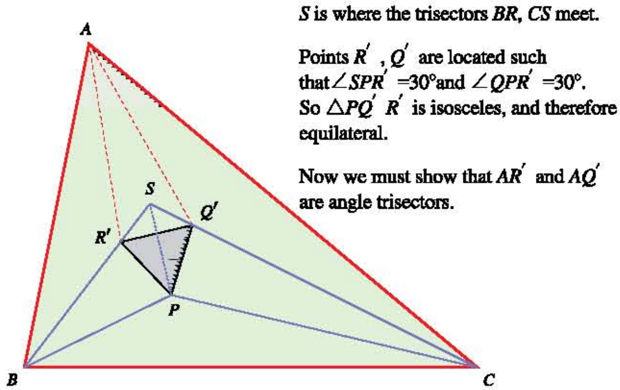

FIGURE 3. Construction and strategy of Naraniengar's proof

To this end, Naraniengar uses the following lemma (see Figure 4):

Lemma. *If four points* $\boldsymbol{M, R', Q', N}$ *satisfy the following conditions:*

(i) $\boldsymbol{MR' = R'Q' = Q'N}$, *(ii)* $\boldsymbol{\angle MR'Q' = \angle R'Q'N = 180^\circ - 2x > 60^\circ}$, *for some value of* $\boldsymbol{x}$ *less than 60°, then the four points* $\boldsymbol{M, R', Q', N}$ *lie on a circle. Further, if point* $\boldsymbol{A}$ *on the side of* $\boldsymbol{MN}$ *away from* $\boldsymbol{R'}$, *is so situated that* $\boldsymbol{\angle MAN = 3x}$, *then* $\boldsymbol{A}$ *lies on the same circle.*

To see why this is true, draw the bisectors of $\boldsymbol{\angle MR'Q'}$ and $\boldsymbol{\angle R'Q'N}$, and let them meet at $\boldsymbol{O}$. Then we have $\boldsymbol{\Delta OMR' \cong \Delta OR'Q' \cong \Delta OQ'N}$ (side-angle-side congruence), hence $\boldsymbol{OM = OR' = OQ' = ON}$. Therefore the four points $\boldsymbol{M, R', Q', N}$ lie on a circle $\boldsymbol{\omega}$ centred at $\boldsymbol{O}$.

Hence $\angle OMR'$, $\angle OR'M$, $\angle OR'Q'$, $\angle OQ'R'$, $\angle OQ'N$ and $\angle ONQ'$ are all equal to $90° - x$. But this implies that $\angle MOR'$, $\angle R'OQ'$ and $\angle Q'ON$ are all equal to $2x$, and hence that $\angle MON = 6x$.

Hence to show that A lies on ω, it suffices to: (i) note that A lies on the same side of MN as O, and (ii) show that $\angle MAN = 3x$.

To use this result we refer to Figure 5 (which is Figure 3, redrawn) and locate points M on side AB and N on side AC such that $BM = BP$ and $CN = CP$. Then $\Delta PBR' \cong \Delta MBR'$, so $MR' = PR'$ and so $MR' = R'Q'$. In the same way we have $NQ' = R'Q'$.

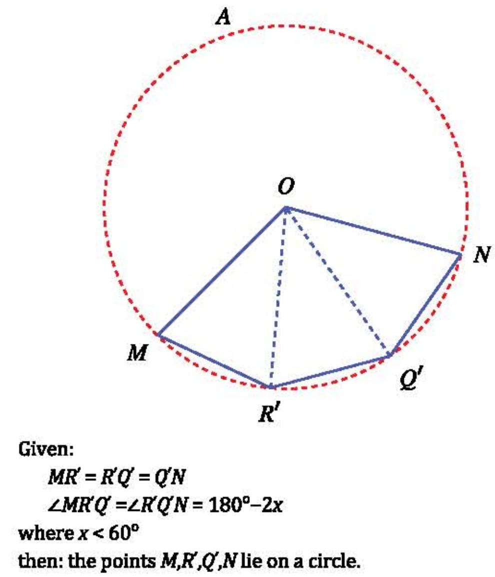

FIGURE 4. Lemma used by Naraniengar

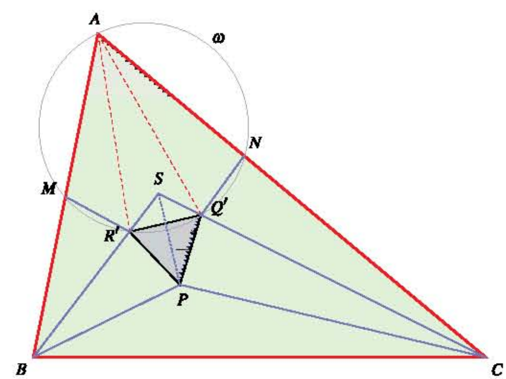

FIGURE 5. Applying the lemma

Now we must compute the measures of $\boldsymbol{\angle MR'Q'}$ and $\boldsymbol{\angle NQ'R'}$.

Let $\boldsymbol{\angle A = 3x}$, $\boldsymbol{\angle B = 3y}$, $\boldsymbol{\angle C = 3z}$. Since $\boldsymbol{\angle BSC = 180^\circ - 2y - 2z = 60^\circ + 2x}$ (remember that $\boldsymbol{3x + 3y + 3z = 180^\circ}$), we get: $\boldsymbol{\angle PSR'} = 30° + \boldsymbol{x}$. Since $\boldsymbol{\angle SPR'} = 30°$, we get $\boldsymbol{\angle BR'P = 60^\circ + x}$ and $\boldsymbol{\angle MR'B = 60^\circ + x}$ as well. Hence $\boldsymbol{\angle MR'Q'} = 360° - (120° + \boldsymbol{2x + 60^\circ}) = 180° - \boldsymbol{2x}$. In the same way we have $\boldsymbol{\angle NQ'R'} = 180° - \boldsymbol{2x}$.

The conditions of the lemma now hold, so we can assert that points $\boldsymbol{A, M, R', Q', N}$ lie on a circle $\boldsymbol{\omega}$ as shown in Figure 4. Since chords $\boldsymbol{MR', R'Q', Q'N}$ of this circle have equal length, they subtend equal angles at $\boldsymbol{A}$, implying that $\boldsymbol{AR'}$ and $\boldsymbol{AQ'}$ are angle trisectors. This now establishes what we set out to prove, and so Morley's theorem is proved.

Remarks. Naraniengar's proof follows an unusual strategy: it *starts* with an equilateral triangle, then sets up a configuration similar to that constructed in the theorem. This clearly implies that the triangle constructed by Morley is equilateral. It is curious that a similar strategy is pursued in many pure-geometry proofs of Morley's theorem: start with an equilateral triangle, then reconstruct a configuration similar to the original one. Perhaps the most spectacular of these is the proof by John Conway. In the next part we examine proofs that use trigonometry.

References

[1] Bankoff, L. "A simple proof of the Morley theorem", Math. Mag., 35 (1962) 223–224

[2] Bogomolny, A. "Morley's Miracle", http://www.cut-the-knot.org/triangle/ Morley/index.shtml.

[3] Child, J. M. "Proof of Morley's Theorem".Math. Gaz. 11, 171, 1923.

[4] Conway, J. H. "Proof of Morley's Theorem", http://www.cambridge2000.com/memos/ pdf/conway2.pdf

[5] Coxeter, H. S. M. "Introduction to Geometry", 2nd ed., Wiley, New York, 1969

[6] Coxeter, H. S. M. & Greitzer, S. L. "Geometry Revisited", Random House/Singer, New York, 1967

[7] Gardner, M. "New Mathematical Diversions from Scientific American", Simon and Schuster, New York, 1966, pp. 198, 206

[8] Honsberger, R. "Morley's Theorem",Mathematical Gems I.Washington, DC:Math. Assoc. Amer., pp. 92-98, 1973.

[9] Johnson, R. A. Advanced Euclidean Geometry, Dover, New York, 1960

[10] Knight, G. H. Morley's theorem, New Zealand Math. Mag., 13 (1976) 5-8.

[11] Weisstein, Eric W. "Morley's Theorem". From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/MorleysTheorem.html.

[12] Naraniengar, M. T. Solution to Morley's problem, "The Educational Times, with many Papers and Solutions in addition to those published in The Educational Times," New Series, 15 (1909) 47

[13] Robson, A. Morley's theorem,Math. Gaz., 11 (1922-1923) 310-311

[14] Stonebridge, B. A Simple Geometric Proof of Morley's Trisector Theorem, http://ms. appliedprobability.org/data/files/selectedarticles/42-1-1.pdf

[15] Wikipedia. "Morley's trisector theorem". http://en.wikipedia.org/wiki/Morley's_ trisector_theorem.

PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1994. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.