feature

In mathematics, breaking up . . .

Equalizers of a Triangle

. . . is not hard to do!

In the March 2014 issue of At Right Angles, the article "A Fair Division" presented a study of a problem involving a geometrical division. A plot of land in the form of a scalene triangle is to be divided, as per the dictates of a whimsical will, into two parts having equal area as well as equal perimeter, using a straight dividing line. A simple argument shows that there always exists such a line; see [2]. In the mathematical literature, such a line has been called the equalizer of the triangle. It is known that any triangle has 1, 2 or 3 equalizers; see [4]. In this article we prove two results related to the equalizers.

Shailesh Shirali

The results mentioned in the preamble above are not only beautiful but remarkable as well, packing a good deal of 'surprise value'. Here they are:

Theorem 1. An equalizer of a triangle necessarily passes through its incentre.

Theorem 2. A line passing through the incentre of a triangle divides its perimeter and area in the same ratio.

Theorem 1 is a known result (see [1], [3], [5]). We have not seen Theorem 2 anywhere in the literature. The proofs of both the theorems are easy to find. We invite you to find your own proofs before reading ahead.

Keywords: *Triangle, perimeter, area, ratio, incentre, equalizer, quadratic, roots*

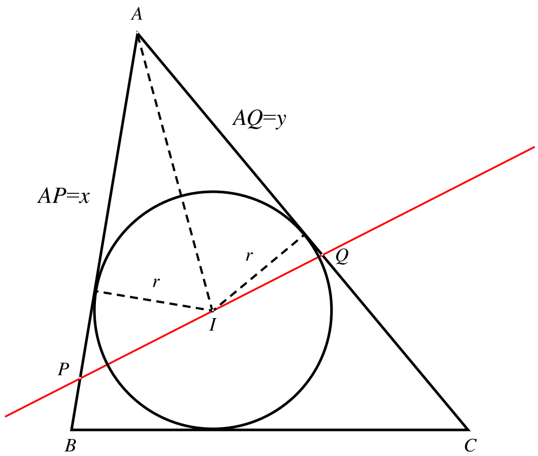

FIGURE 1

Figure 1

Proof of Theorem 2. The two results are proved in nearly the same way, but we choose to prove Theorem 2 first. Figure 1 shows a triangle ABC with incentre I, with an arbitrary line ℓ through I. This line must pass through some two sides of the triangle, and we shall suppose them to be AB and AC. Let the points of intersection of ℓ with AB and AC be P and Q respectively; let $AP = x$ and $AQ = y$. The theorem then claims the following:

$$\frac{\text{Area of } \triangle APQ}{\text{Area of quadrilateral } PBCQ} = \frac{AP + AQ}{PB + BC + CQ}.$$

This may be written in the following equivalent form:

$$\frac{\text{Area of } \triangle APQ}{\text{Area of } \triangle ABC} = \frac{AP + AQ}{AB + BC + CA}.$$

Let $2s = a + b + c$ be the perimeter of $\triangle ABC$. Then we must show the following:

$$\frac{\text{Area of } \triangle APQ}{\text{Area of } \triangle ABC} = \frac{x + y}{2s}.$$

Consider the fraction on the right side. Multiplying both the numerator and denominator by the in-radius r, we get the following:

$$\frac{x+y}{2s} = \frac{r(x+y)}{2rs} = \frac{\frac{1}{2}rx + \frac{1}{2}ry}{rs}.$$

In the last expression, note that $\frac{1}{2}rx$ is the area of $\triangle API$ (because if we treat $x = AP$ as the base, then its altitude is r) and, similarly, $\frac{1}{2}ry$ is the area of $\triangle AQI$. Hence $\frac{1}{2}rx + \frac{1}{2}ry$ is the area of $\triangle APQ$. Also, rs is the area of $\triangle ABC$. (This is a known formula. To prove it, note that the area of $\triangle ABC$ is the sum of the areas of $\triangle IBC$, $\triangle ICA$ and $\triangle IAB$. Now treat BC, CA and AB as the bases of these triangles, and note that all three triangles have the same altitude, r; now fill in the rest of the proof.) Hence the expression is equal to the ratio

$$\frac{\text{Area of } \triangle APQ}{\text{Area of } \triangle ABC}.$$

But that is just what we wanted to show! Hence, Theorem 2 is proved.

Proof of Theorem 1. We adopt a very similar strategy. Let the line ℓ bisect the perimeter as well as the area of $\triangle ABC$. As earlier, we argue that ℓ must intersect some two sides of the triangle; let them be AB and AC, and let the points of intersection of ℓ with these two sides be P and Q respectively. Let $AP = x$ and $AQ = y$.

The fact that ℓ is an equalizer implies that $x + y = s$ and $xy = \frac{1}{2}bc$. Let the internal bisector of $\angle BAC$ meet ℓ at J. We must then show that J is the incentre of $\triangle ABC$. (See Figure 2.)

From J, drop perpendiculars JU and JV to AB and AC respectively. Since J lies on the bisector of $\angle A$, it follows that $JU = JV$; let their common length be r'. To show that J is the incentre of $\triangle ABC$ is equivalent to showing that r' equals the in-radius r of $\triangle ABC$, and this is what we shall now show.

The areas of $\triangle AJP$ and $\triangle AJQ$ are $\frac{1}{2}r'x$ and $\frac{1}{2}r'y$ respectively, so the area of $\triangle APQ$ is $\frac{1}{2}r'(x+y)$. Since $x + y = s$, it follows that the area of $\triangle APQ$ is $\frac{1}{2}r's$. But since ℓ is an equalizer, the area of $\triangle APQ$ is half the area of $\triangle ABC$; hence the area of $\triangle ABC$ is $r's$. But the area of $\triangle ABC$ is also equal to rs. It follows that $r' = r$ and hence that J is the incentre of the triangle. Thus the equalizer passes through the incentre of the triangle, as claimed.

Locating the Equalizers. A candidate line ℓ for the post of equalizer of a triangle ABC must pass through some two sides of the triangle, say *AB & AC. Let* ℓ cut these two sides at P and Q respectively,

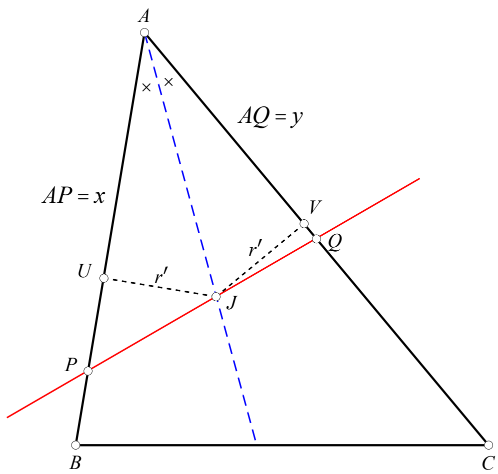

Figure 2

and let $AP = x$, $AQ = y$. As ℓ is an equalizer, we have $x + y = s$ (where $2s = a + b + c$ is the perimeter of the triangle) and $xy = \frac{1}{2}bc$. Hence an equalizer passing through sides AB and AC exists if and only if the equations $xy = \frac{1}{2}bc$, $x + y = s$ yield values for x, y satisfying the inequalities $0 \le x \le c$ and $0 \le y \le b$.

Now if $x, y \ge 0$ and $x + y = s$, the range of possible values of xy is $0 \le xy \le \frac{1}{4}s^2$; the least possible value is taken when one of x, y is 0, and the maximum possible value is taken when $x = y = \frac{1}{2}s$ (because if the sum of two numbers is held fixed, their product is largest when the numbers are equal). For a solution to exist, a *necessary* condition is that $\frac{1}{2}bc$ lies within this interval. So we must have $\frac{1}{2}bc \le \frac{1}{4}s^2$, i.e., $s^2 \ge 2bc$. If this inequality is strict, there is a possibility of two solutions (x, y), while if equality holds ($s^2 = 2bc$), there is just one solution. Note that we say 'possibility' —because we also need the inequalities $0 \le x \le c$ and $0 \le y \le b$ to hold (for P, Q must lie on sides AB, AC respectively). The actual values of x, y (got by solving the equations $x + y = s$, $xy = \frac{1}{2}bc$) are:

$$x, y = \frac{s \pm \sqrt{s^2 - 2bc}}{2}.$$

Case study–I: Triangle with sides 3, 4, 5 We take each pair of sides in turn to be candidates for $\{b, c\}$, and check for feasible solutions. Here $s = 6$, so $s^2 = 36$.

- $\{b, c\} = \{3, 4\}$. Here $2bc = 24$, so $s^2 > 2bc$. Solving for x, y, we get:

  $$x, y = \frac{6 \pm \sqrt{36 - 24}}{2} = 3 \pm \sqrt{3}$$

  Neither choice of sign works, because $3 + \sqrt{3} > 4$. So we do not get any equalizer associated with this pair of sides.

- $\{b, c\} = \{3, 5\}$. Here $2bc = 30$, so $s^2 > 2bc$. Solving for x, y, we get:

  $$x, y = \frac{6 \pm \sqrt{36 - 30}}{2} = 3 \pm \sqrt{1.5}$$

  Since $3 - \sqrt{1.5} < 3$ and $3 < 3 + \sqrt{1.5} < 5$, we get one equalizer here (but only one).

- $\{b, c\} = \{4, 5\}$. Here $2bc = 40$, so $s^2 < 2bc$. This does not yield any equalizers.

So for the 3, 4, 5 triangle, there exists just one equalizer. Figure 3 gives a sketch of the situation. The sole equalizer PQ has been drawn, with $CP = 3 - \sqrt{1.5}$ and $CQ = 3 + \sqrt{1.5}$. The equalizer passes through the incentre I, as it should. Observe that $CP + CQ = 6 = s$, and $CP \times CQ = 9 - 1.5 = 7.5 = \frac{1}{2}(3 \times 5)$.

Case study–II: Triangle with sides 7, 8, 9
As earlier, we take each pair of sides in turn to be candidates for $\{b, c\}$, and check for feasible solutions. Here $s = 12$, so $s^2 = 144$.

- $\{b, c\} = \{7, 8\}$. Here $2bc = 112$, so $s^2 > 2bc$. Solving for x, y, we get:

  $$x, y = \frac{12 \pm \sqrt{144 - 112}}{2} = 6 \pm 2\sqrt{2}.$$

  Neither choice of sign works, because $6 + 2\sqrt{2} > 8$. So we do not get any equalizer associated with this pair of sides.

- $\{b, c\} = \{7, 9\}$. Here $2bc = 126$, so $s^2 > 2bc$. Solving for x, y, we get:

  $$x, y = \frac{12 \pm \sqrt{144 - 126}}{2} = 6 \pm \sqrt{4.5}.$$

  Since $7 < 6 + \sqrt{4.5} < 9$, we get one equalizer (but only one).

- $\{b, c\} = \{8, 9\}$. Here $2bc = 144$, so $s^2 = 2bc$. Solving for x, y, we get:

  $$x, y = \frac{12 \pm \sqrt{144 - 144}}{2} = 6.$$

  Since $6 < 9$, we get an equalizer here. Since $x = y$ in this case, the two equalizers are coincident.

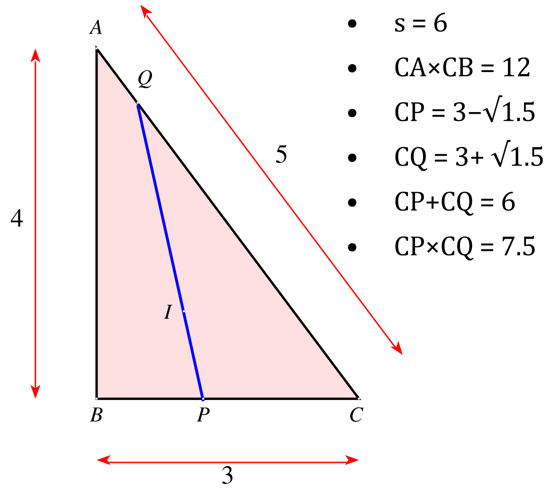

Figure 3. Equalizer for a 3,4,5 triangle;
I is the incentre (there is just one equalizer)

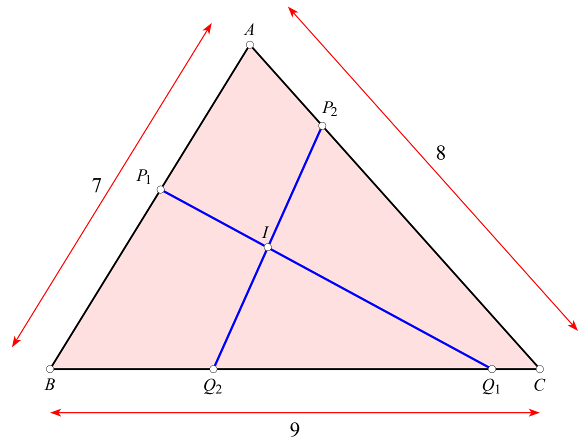

Figure 4. Equalizers for a 7, 8, 9 triangle: P_1Q_1, with $BP_1 = 6 - \sqrt{4.5}$ and $BQ_1 = 6 + \sqrt{4.5}$; P_2Q_2, with $CP_2 = 6 = CQ_2$; I is the incentre

So for the 7, 8, 9 triangle, there exist two equalizers. Both of them have been sketched in Figure 4 (segments $P_1\,Q_2$ and $P_2\,Q_2$).

An equilateral triangle obviously has three equalizers (all three medians). So we may anticipate that as the triangle changes in shape from a high degree of scalene-ness towards equilateral-ness, the number of equalizers changes from 1 to 3. A complete analysis of how this change happens is given in [4]. However, we do not try to prove this here.

References

[1] CTK Insights, "Area and Perimeter Splitters in a Triangle", http://www.mathteacherctk.com/blog/2013/05/area-and-perimeter-splitters-in-a-triangle/

[2] P. De & S. Titus & S. Sircar, "A Fair Division", *At Right Angles,* APU, March 2014

[3] R. Honsberger, *Mathematical Delights*, MAA, 2004, 71–74

[4] D. Kodokostas, "Triangle Equalizers", *Mathematics Magazine* **83**, No. 2 (April, 2010) 141–146

[5] S. Kung, "Proof Without Words: A Line through the Incenter of a Triangle", *Mathematics Magazine* **75**, No. 3 (Jun., 2002) 214

SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as an editor for Resonance and At Right Angles. He may be contacted at shailesh.shirali@gmail.com.