in the classroom

Flashback to the past

A 1949 Matric Geometry Question

Michael De Villiers

Recently I was paging through my copy of the August 1996 issue of the unfortunately now defunct journal *Spectrum 34(3)*, and a section on old Mathematics Papers on p. 63 where the following problem from Paper 2 of the 1949 National Senior Examinations for the Union of South Africa caught my attention:

"In a quadrilateral *ABCD*, angles *B* and *C* are right angles. A straight line *EF* is drawn perpendicular to *AD*, and cuts *AD* and *BC* internally at *E* and *F* respectively. Prove that ∠*BEC* = ∠*AFD*."

Before continuing further, the reader is now urged to first try and prove the result.

For those of a younger generation, it may come as a surprise that like this question, none of the geometry questions included a sketch or diagram, as is customary in matric exams today. At the time it was expected of pupils (learners) to read, interpret and be able to draw their own diagrams from such a verbal, symbolic description. Even when I was in high school in the early seventies, this was still the case, but this was changed some time in the late seventies or early eighties to probably try and make things easier, not only for pupils (learners), but probably also for examiners in not having to decipher pupils' (learners') rough drawings when marking.

It should be noted, however, that in high-level mathematical competitions such as the Senior Third Round of the South African Mathematical Olympiad (SAMO) as well as the International Mathematical Olympiad (IMO), it is still customary that no diagrams are provided for learners, and that they are required to make their own drawings. So at this level, interpreting a verbal description of a geometric problem, and making an appropriate representation, is seen as part and parcel of mathematical competence and creativity. See for example, some past SAMO and IMO papers at the following two websites respectively:

http://www.samf.ac.za/QuestionPapers.aspx
http://www.imomath.com/index.php?options=924&lmm=0

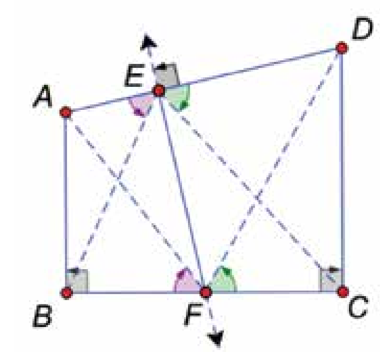

Figure 1

Proof

The proof is no easier or more difficult than ones currently expected from Grade 11/12 learners. Consider Figure 1. From the given construction, it follows that *ABFE* and *DCFE* are both cyclic quadrilaterals, since ∠*AEF* = ∠*DCB* and ∠*DEF* = ∠*ABC*, respectively. Therefore, on chord *AB*, ∠*AEB* = ∠*BFA* and on chord *DC*, ∠*DEC* = ∠*CFD*. It now follows that, ∠*BEC* = 180° – ∠*AEB* - ∠*DEC* = 180° – ∠*BFA* – ∠*CFD* = ∠*AFD*, which completes the proof .

Can we prove it differently? As is most often the case, another easy way to prove the result comes from proving the similarity of triangles *AFD* and *BEC* (since corresponding ∠*EAF* = ∠*EBF* on chord *EF* in cyclic *ABFE* and ∠*EDF* = ∠*ECF* on chord *EF* in cyclic *DCFE*). Note that we often prove mathematical theorems in different ways, not because we feel a need to verify the validity of the result, but *merely to increase our understanding or to look at it in a* different way (De Villiers, 1990).

Further Reflection

Pólya (1945) has most famously emphasized that the fourth step in problem solving is that of looking back, and reflecting on one's solution. Unfortunately this aspect of problem solving is seldom highlighted in teaching: usually the teacher and the children are in a hurry and just happy to move on to the next problem. However, reflecting on a solved problem or proof can often lead not only to deeper understanding and appreciation of a particular result, but also show how mathematics is sometimes developed further.

First, note that the result is really about a trapezium *ABCD* with *AB* and *DC* parallel (both perpendicular to *BC*). Secondly, looking back at the proof, it should be clear that the result depends on *ABFE* and *DCFE* both being cyclic quadrilaterals. But is it really necessary that the angles at *B* and *C* are right angles?

Clearly not, as all that is required is that the following remains true: ∠*AEF* = ∠*DCB* and ∠*DEF* = ∠*ABC*. Since ∠*AEF* + ∠*DEF* = 180°, it follows that the only requirement is that ∠*DCB* + ∠*ABC* = 180°; in other words, that *AB* has to be parallel to *DC*. In other words, the result is true for any trapezium *ABCD* with *AB* // *DC* and line EF constructed so that ∠*AEF* = ∠*DCB* (or equivalently ∠*DEF* = ∠*ABC*), as shown in Figure 2. It is left to the reader to verify that exactly the same proof applies. This is therefore an example of what has been called the *discovery* function of proof (De Villiers, 1990), whereby further analyzing the conditions of a proof can sometimes lead to further generalizations.

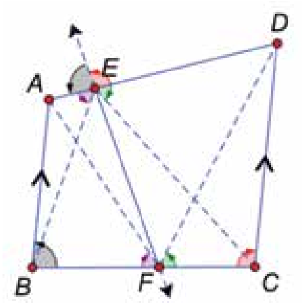

Figure 2

What if?

'Doing mathematics' is not just about answering questions, but also about asking questions, investigating and posing one's own problems. In this regard, an important mathematical habit of the mind is to ask 'what-if?' questions. Though the original problem was restricted to *E* and *F* on the 'interior' of segments *AD* and *BC* respectively, an obvious 'what-if' question to ask is: what happens when these points lie on the extensions of *AB* and *BC*? The reader is now invited to dynamically explore this situation with the following interactive sketch online:

http://dynamicmathematicslearning.com/matric-exam-1949.html

As the reader would've found, the equality of the two angles remains true even when *E* and *F* are moved outside onto the extensions of segments *AB* and *BC*. The result follows just as before, but with some variation. Consider, for example, the situation as shown in Figure 3 where both *E* and *F* lie on the extensions of the two segments.

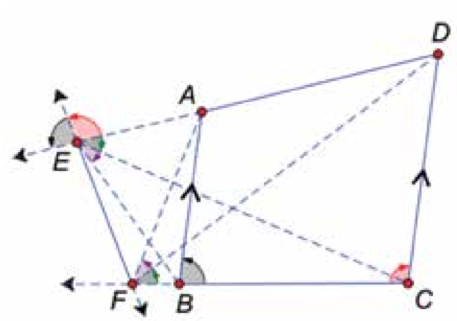

Figure 3

As before, it follows that *ABFE* and *DCFE* are both cyclic quadrilaterals, since $\angle AEF = \angle ABC$ and $\angle DEF + \angle DCF = 180^o$, respectively. Therefore, on chord *AB*, $\angle AEB = \angle BFA$ and on chord *DC*, $\angle DEC = \angle CFD$. It now follows that $\angle BEC = \angle AEB - \angle DEC = \angle BFA - \angle CFD = \angle AFD$, which completes the proof. It's now left to the reader to verify that the result also holds for the other cases if *E* and *F* lie on the extensions of *AD* and *BC* towards the opposite sides, or when one of *E* or *F* lies on a segment while the other lies on an extension.

Since there are several cases to consider, it is probably for this reason that the examiners decided to simplify the problem by restricting it to the interior of the trapezium, but in the process the interesting generality of the result was unfortunately lost. It should be noted, however, that if one uses the advanced concept of 'directed angles', or better still, use vector or complex algebra, it is possible to give a general proof that covers all cases (including the crossed case discussed below).

Though the examiners probably only had a convex trapezium in mind (assuming perhaps implicitly that *BA* and *CD* were on the same side of *BC*), another configuration not explicitly excluded by the examiners is the case when the trapezium becomes crossed as shown in Figure 4. Does the result still hold for the crossed trapezium *ABCD* shown?

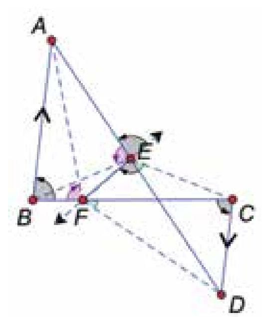

Figure 4

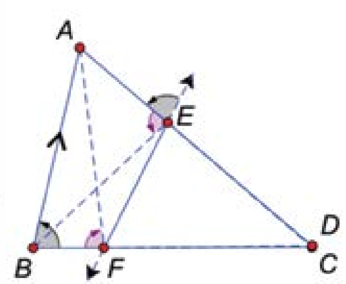

Figure 5

As can easily be verified experimentally by the reader, the result is still valid. For example, simply drag the point *D* in the dynamic geometry sketch in the link given earlier, in the opposite direction until it moves past *C* as shown in Figure 4,

However, the previous proofs require a little modification for the crossed case. As before, *ABFE* is still cyclic since the exterior $\angle FED = \angle ABF$ (by construction). Since *AB* // *CD*, alternate angles *ABF* and *FCD* are equal; and $\angle FED = \angle FCD$; hence *EFDC* is also a cyclic quadrilateral (equal angles on chord *FD*). Therefore, as before, on chord *AB*, $\angle AEB = \angle BFA$, and on chord *DC*, $\angle DEC = \angle CFD$. It now follows that:

$\angle BEC = 180^o - \angle AEB + \angle DEC = 180^o - \angle BFA + \angle CFD = \angle AFD$

Also note, as shown in Figure 5, that angles *BEC* and *AFD* remain equal in the special case when, say, *C* and *D* coincide; in other words when the trapezium degenerates into a triangle.

Alternative formulations

Since the problem involves cyclic quadrilaterals, we could instead of starting with a trapezium, consider what happens if we start with two intersecting circles *P* and *Q* as shown in Figure 6. If we construct straight lines *AED* and *BFC* through the two intersections of the circles as shown, we obtain *AB* // *CD*, and as before $\angle BEC = \angle AFD$. In a sense, this variation can be considered as a type of converse of the original result since the premise here is the intersecting circles (i.e. implicitly the cyclic quadrilaterals *ABFE* and *EFDC*) with the conclusion now that *ABCD* is a trapezium with *AB* // *CD* (whereas in the original general version formulated with respect to Figure 2, this conclusion is assumed as given). It is left to the reader to verify that the result holds in the case illustrated (as well as other possible configurations such as a crossed configuration). Another interesting converse-like formulation of the result is the following. Given the two circles,

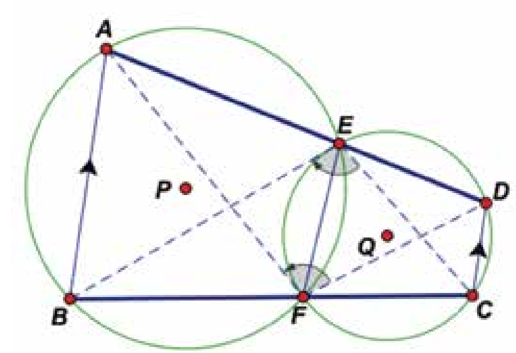

Figure 6

if *AB* is drawn first, followed by the straight line *BFC*, and *CD* is then drawn parallel to *BA*, then *D*, *E* and *A* are collinear (lie in a straight line). The proof of this is similar to those before, and is also left as an exercise to the reader.

Maximizing the angle

If the reader goes back to the original dynamic sketch, and drags the moveable point *E*, it will be immediately noticeable that $\angle BEC$ is a variable. This now raises another interesting question, namely, whether the angle has a maximum and where to locate it.

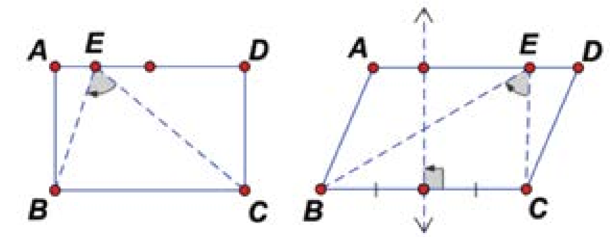

Figure 7

As is well known in problem solving, it often helps to understand a problem better by looking at special cases. If we consider a rectangle as a special case of a trapezium as shown in the first figure in Figure 7, it is obvious from symmetry that the maximum value of angle *BEC* will be at the midpoint of *AD*. In the case of the trapezium being a parallelogram, it's not hard to intuitively 'see' (or find experimentally with a dynamic geometry sketch) that the optimal value of angle *BEC* is obtained when *E* is located at the intersection of the perpendicular bisector of *BC* and the opposite side *DA*.

What about the general case when *ABCD* is a trapezium with *AB* // *CD*? Remembering the analogous problem of maximizing the kicking angle in rugby as discussed in De Villiers (1999), it was clear to me that the solution would similarly lie where the circle through *B* and *C* touched the line *AD*.

Consider Figure 8. It is now easy to prove as it follows that angle *BEC* would be maximized when *E* is placed at the tangent point, *G*, where the circle through *B* and *C* touches the line *AD*. Since they are on the same chord *BC*, any inscribed $\angle BHC = \angle BGC$. But from the exterior angle theorem, it follows that $\angle BHC > \angle BEC$; hence $\angle BEC$ is a maximum only when *E* is placed at *G*.

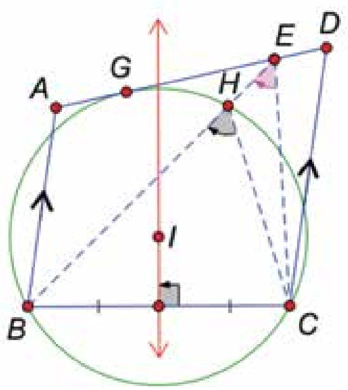

Figure 8

One can easily find this tangent point *G* experimentally using dynamic geometry by first constructing the perpendicular bisector of *BC*. Then locating a centre *I* of the circle on the perpendicular bisector, one can move the point *I* up and down until the circle touches the line *AD*. However, this is not very satisfactory as apart from being prone to 'experimental error', it is dependent entirely on a particular trapezium *ABCD*, and as soon as the trapezium's shape is changed, one will have to repeat the same experimental approach of moving the circle. So ideally, we'd like to have a way of precisely locating the centre *I* of the tangent circle that gives the optimal position of *E*. How can that be done?

Think about it! We are looking for a point that is equidistant from the points *B* and *C* as well as the line *AD*. Since a perpendicular bisector is the locus of all equidistant points from *B* and *C*, the point *I* clearly must lie somewhere on the perpendicular bisector of *BC*, but exactly where?

Recall a result already known to the ancient Greeks that the locus of all points equidistant from a point (called a focus) to a line (called the directrix) is a parabola[1]! Therefore, all we now need to do is use a standard dynamic geometry facility to construct the parabola determined by point *B* as focus and line *AD* as directrix, and where this parabola meets the perpendicular bisector, we have the required point *I* that is equidistant from line *AD* as well as points *B* and *C* as shown in Figure 9. *Voila!*

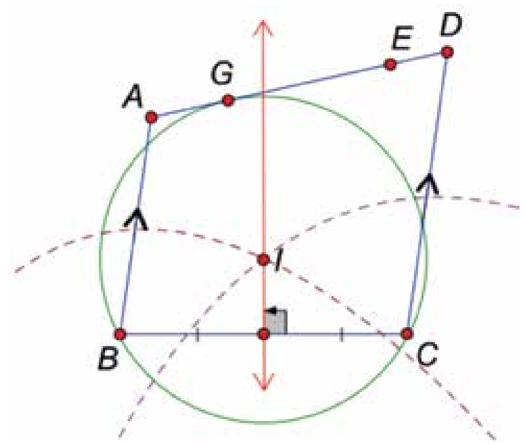

Figure 9

Also note that from the construction above, since *I* is equidistant from points *B* and *C* as well as line *AD*, the point *I* also lies on the parabola determined by *C* as focus and *AD* as directrix. Hence this also proves the interesting, unusual result of two parabolas and a line concurrent in a point, something which learners at high school are usually not exposed to. Also note that since the result only involves the line *AB* and the points *B* and *C*, it is not specific to a trapezium any more, but applies to determining the maximum of ∠*BEC* in any quadrilateral *ABCD* with *E* on line *AD*. With the increasing availability of dynamic geometry software in classrooms around the world, this general problem and result can easily be tackled and solved at high school.

In addition, the optimization problem about the angle is more elegantly solved without the use of calculus, purely with synthetic geometry. It can therefore show high school learners the value of geometry, and that one need not always resort to algebra and calculus to minimize or maximize a particular variable.

[1]*For more information on parabola, go to: https://en.wikipedia.org/wiki/Parabola Or see my paper "Exploring loci on Sketchpad" at: http://www.researchgate.net/publication/242188586_Exploring_Loci_on_Sketchpad*

Concluding comments

Starting with a standard matric (Grade 12) problem from years ago, which on the surface may even appear rather routine and boring, this paper has shown how further reflection and exploration of the original result has actually revealed a very rich problem leading to generalizations, alternative formulations and an optimization problem. Sometimes exploring one problem in more depth like this one can indeed be a better educational experience for learners than doing numerous examples, but only covering them superficially.

References

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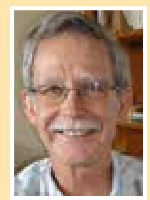

MICHAEL DE VILLIERS has worked as a researcher and mathematics and science teacher at several institutions across the world. Since 1991 he has been part of the University of Durban-Westville (now the University of KwaZulu-Natal). He has already numerous books and articles. He was editor of *Pythagoras*, the research journal of the Association of Mathematics Education of South Africa (AMESA) and has been vice-chair of the SA Mathematics Olympiad since 1997. He is a regular speaker at conferences on mathematics and math education. His main research interests are Geometry (especially Dynamic Geometry), Proof, Applications and Modeling, Problem Solving, and the History and Philosophy of Mathematics. His home page is http://dynamicmathematics-learning.com/homepage4.html. He maintains a web page for dynamic geometry sketches at http://dynamic-mathematicslearning.com/JavaGSPLinks.htm. He may be contacted on profmd@mweb.co.za.