in the classroom

No dead-ends

Learning mathematics through puzzles

No solution is also a good solution

Mathematical puzzles are generally perceived to be at the periphery of mathematics, and not part of the core of the discipline. This may be related to the fact that we sometimes come across puzzles that have no solution. The general expectation among mathematics practitioners and school children seems to be that problems in mathematics should have a solution. In this paper, we argue that puzzles can be an important source of learning some core mathematical ideas. We present an exemplar to justify our claim that problems with no solutions are as significant to the teaching and learning of mathematics as problems with solutions.

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Introduction

Often puzzles are considered as games or 'time-off' from routine mathematics in school classrooms. The objective is generally to find who solves the puzzle, how quickly and what is the shortest and probably the quickest way of solving it. But despite knowing that they are mathematical in nature, they are not seen as 'connected with hard core mathematics'.

Many of these puzzles contain all the ingredients and themes of mathematics: proof, generalization, pattern recognition, assuming the truth of a statement and arriving at a contradiction, non-existence of a solution, etc. In this article, we present a puzzle which was used to discuss some core mathematical ideas with participants at two public programs. The participants ranged from fourth standard students to B Ed graduates. For these programs, we designed a few puzzles which required unconventional methods of problem solving.

Keywords: *Puzzle, game, solution, parity, invariance, proof, validity*

By ‘unconventional’ we mean that solving these puzzles was not necessarily confined to elementary mathematics, a well defined syllabus, or an algorithmic solution. Central to our considerations was also that these puzzles should be able to create an opportunity for students to learn some important concepts in mathematics.

Puzzle: Circles and counters

In [1, p. 125] we came across the following puzzle.

A circle is divided into six sectors and each of them has a counter or button in it (see Figure 1). *You have to get all the counters into one sector using jumps, following these two rules:* (*i*) *In a single jump, a counter can be moved only to an adjacent sector.* (*ii*) *Each move consists of two jumps.*

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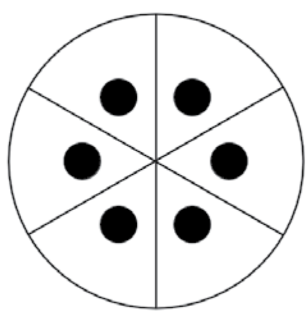

Figure 1. Circle divided into 6 sectors with a counter in each sector

The focus of this puzzle is on parity and invariance, and the solution proposed is that *there is no solution*. The proof for this assertion goes as follows.

Number the sectors from 1 to 6. For each sector, find the product of its number and the number of counters in it. Let the sum of these products be s. Call this sum the ‘score’. At the start of the game, each sector has one counter, so the score is $1 + 2 + 3 + 4 + 5 + 6 = 21$.

After a move in which one counter jumps from, say, 2 to 3 and the other counter jumps from, say, 4 to 5, the new score would be $1 + 0 + (2\times3) + 0 + (2\times5) + 6 = 23$. Note that the score has changed by 2. A little thought will reveal that no matter how the moves are made, the score always changes by an even number; more specifically, by 0, 2, 4 or 6.

Since the starting value of s is 21, an odd number, the value of s will be odd at every stage. But if all the counters reach one sector, the value of s will surely be a multiple of 6 at that stage, and hence even. Therefore the puzzle can have no solution.

We found it interesting that the puzzle did not have a solution. Then we wondered how children and teachers would look at a mathematical problem like this which does not have a solution. Often, such problems are rejected as being wrong or as having inadequate information. However, realising that “no solution is also a valid solution”, mathematically, is interesting and important for teachers and students to know.

Also, we can assert that a problem does not have a solution only when we have a clear proof for such a claim.

These are both important mathematical ideas: first, that “no solution” is a valid state of affairs in mathematics, and second, the need for a proof for the claim that a problem does not have a solution.

Both these matters are currently not a part of our understanding of teaching mathematics especially at the elementary level. Proofs are considered to be abstract and are introduced very late in the school curriculum. Also, problems with no solution or multiple solutions are rarely discussed.

As we were working with the puzzle with these considerations in mind, we started thinking about the different variables involved and how changing them would change the problem. The variables included the number of counters and its positions and of jumps.

For example, in the puzzle, there is just one counter in each sector. What if we have two or more counters instead? What if the number of jumps in each move is increased?

When we worked through these variations we found neither of them so interesting. Let us explain why. Placing more counters in each sector does not make it more challenging but rather a tiring exercise. Also, this would not be a suitable extension for very young children. (We were targeting young children because they had not yet been exposed to ideas of ‘proof and proving’.) So we were doubtful whether this would make the puzzle more engaging.

For the second, we found that number of jumps can be either odd or even. The *odd* number of jumps will make it similar to moving in jumps of 1, and *even* would be in terms of jumps of 2. There would be nothing puzzling here!

However, there was another extension we found to be more interesting: varying the *number of sectors* in the circle. We tried to see if there is a way to complete the task for a circle divided into n sectors, for n between 2 and 10. Then we looked for a pattern. While working on the extended puzzle, we noted some interesting mathematical processes. This included playing and identifying a generalised pattern, finding the values of n for which the solution exists and for which a solution does not exist, finding proofs in each case, etc. This is where we experienced all the ingredients of mathematical engagement.

Modified problem and students' solutions

We posed the modified problem to students of different age levels. We then examined their strategies. The problem posed was:

A circle is divided into n sectors and each of them has a counter or button. You must get all of them into a single sector using jumps, following these rules: (i) In a single jump, a counter can be moved only to an adjacent sector. (ii) Each move consists of two jumps. For which values of n can we get all the counters into a single sector? Why do you say so?

Being able to interact with a large group of students from diverse classrooms was a great advantage, and we got this opportunity from two fora.

One occasion was the 'National Science Day' at the Homi Bhabha Centre for Science Education, Mumbai, and the other was a session in the popular lecture series called 'Chai and Why'. (This is an outreach public activity conducted by TIFR. It consists of talks by members of TIFR and is held every second and fourth Sunday of each month at Prithvi Theatre, Juhu, and at Ruparel College, Matunga, Mumbai respectively. The aim of 'Chai and Why' is essentially to popularize mathematics and science.)

, Images from 'Chai and Why

We interacted with a wide range of participants, from children to adults (including mathematicians, physicists, etc.). We were keen to see what kinds of proofs would come forth from learners of different age groups. We share two interesting and representative solutions.

Given enough time, almost all students, even those in grade 4, came up with a generalised pattern. They figured out that a solution to the problem exists for all odd numbers and for numbers divisible by 4. But many of them could not state confidently that a solution does not exist for even numbers not divisible by 4, such as 10. Also, when asked how they could say that a solution exists for all numbers divisible by 4, most students just stuck to examples with smaller numbers. Their general way of solving the problem was trial and error: looking for a solution for $n < 10$, then stating the generalisation. None of them could explain comfortably why a solution does not exist for $n = 2$ and $n = 6$.

One student came close to finding a proof. He came up with the following observation for what made $n = 6$ different from the rest. He said: "Suppose we choose one sector (the target) as the place where all buttons must end up. This implies that the button on that sector requires 0 jumps to reach there. Similarly, the buttons in its (two) adjacent sectors require 1 jump each; the buttons in the sectors adjacent to those require 2 jumps each; etc. Working this way we find that the total number of jumps required to reach the final sector is odd for $n = 6$ or 2, but even for other values of n." Of course, the sector that requires 1 jump to reach the final sector would require 5 jumps if moved in the opposite direction. But this maintains the parity of the total. ('Parity' refers to whether an integer is odd or even. Note that adding an even number to an integer maintains its parity, and adding an odd number reverses its parity. 'Parity invariance' is a commonly used theme in solving problems and in constructing proofs.)

Although he stopped there, it turns out that his observation was a good starting point for a legitimate proof; given more time, he would probably have completed it. To understand and extend his proof, we continue the argument. For odd n, if a sector requires an odd number of jumps to reach the target in one direction, it requires an even number of jumps if moved in the opposite direction. Hence we can always end up in the target sector by choosing for each button an appropriate direction so as to ensure an even number of jumps. For n divisible by 4, the sector diametrically opposite the target requires an even number of jumps to reach the target. The remaining sectors are symmetrically placed. Thus, every such sector has a corresponding sector with the same number (of jumps to reach the target).

Our proof

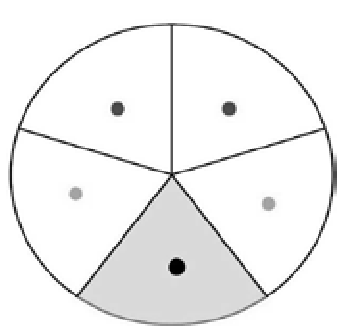

Figure 2. Circle divided into $\boldsymbol{n} = 5$ sectors with bottom sector as 'target'

Our proof was on similar lines as that of the student. Let n be the number of sectors of the circle. Then n is either odd, or divisible by 4, or even but not divisible by 4.

The case when n is odd: Let us choose one sector as the target (see Figure 2). Now we have an even number of buttons to be moved to that sector. These sectors are placed symmetrically and can thus be moved to the target in steps of two jumps: one jump of a button towards the target, followed by another jump by its corresponding button in a symmetric manner. The status of the puzzle after the first move is shown in Figure 3. Therefore, we can always end with all the buttons in a common sector if n is odd.

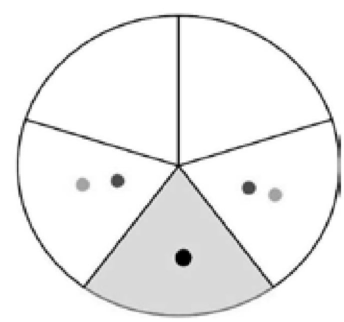

Figure 3. State of the puzzle for $n = 5$ after one move (i.e., two jumps)

The case when *n* is divisible by 4: Designate one sector as the target. Since n is a multiple of 4, the counter diametrically opposite to the target would require, and can be moved by, an even number of jumps to reach there. Now again, we are left with an even number of buttons placed symmetrically. These can be moved to the target using the same method as described above for the case of the circle divided into an odd number of sectors. Therefore we have a solution for n divisible by 4 (similar to the student's proof).

The case when *n* is even but not divisible by 4: For $n = 6$, we cannot use the above argument. Of course, this is not a proof. To prove that we cannot have a solution for such values of n, we assign a number to each sector, in the following way. Beginning with any sector, we number them 0 and 1 in alternation. Since n is even, this is possible; further, each '1' will only have a '0' as its neighbour, and vice versa.

Now for each sector we find the product of its number and the number of counters in it, and find the sum (s) of the products. At the start, since each of the sectors has one counter, s would be 1+0+1+0+1+0+... = $n/2$, an odd number. With each move, we make two jumps — in each jump, a button would move from 0 to 1 or from 1 to 0. So, every jump changes s from an even number to an odd number or the reverse; that is, *it reverses its parity.*

Hence two jumps maintain the parity. Since the number of sectors is not divisible by 4, we have an odd number of zeros and an odd number of ones. Hence we begin with s being odd. The parity stays invariant. However, a solution to the problem would imply that all buttons come to a common sector thereby making s even. Therefore a solution does not exist.

Implications of this activity

The aim of the activity was to use 'concept based puzzles' to create challenges that would encourage the development of formal proof among children.

The fact that some students articulated justifications behind the non-existence of a solution gave us some evidence that mathematical puzzles can drive students to perform problem solving activities that are consistent with the nature of mathematics.

The process of drawing upon a puzzle to identify important mathematical ideas and using them to create the need for proof was interesting and insightful. We are finding it possible to engage even very young children in the idea of proof, and engaging them with the centrality of proof in mathematics. The scope for learners to come up with their own proofs would create a legitimate participation in the culture of doing mathematics and such an environment can make them appreciate the significance of rigour in mathematics.

References

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