in the classroom

Math Exotica …

From magic squares to magic carpets

A. Ramachandran

Magic squares are a topic of interest to mathematicians, puzzlers and lay people alike. Apart from the mathematical properties, mystical qualities are often attributed to these in different cultures.

Magic squares are arrays of numbers (usually from 1 onwards) whose rows, columns and diagonals add up to the same 'magic' total. There is essentially just one 3 x 3 magic square with a magic sum of 15, but one could have obtained others by reflections and rotations.

There are a large number of different 4 x 4 magic squares (even excluding reflections and rotations.) In several of these, apart from the rows, columns and diagonals yielding the magic sum of 34, many other symmetrically located quartets of numbers give the same total.

Some 4 x 4 magic squares have the property that pairs of numbers symmetrically placed about the centre of the grid add up to 17. Two such pairs of numbers would form an interesting pattern, yielding the magic sum of 34. Let us for short refer to this property as the 'inversion symmetry'.

Keywords: *Magic squares, pattern, symmetry, inversion, quadrilaterals*

Source: http://en.wikipedia.org/wiki/File:Melencolia_I_(Durero).jpg

One such 4 x 4 magic square features in a celebrated work of art – an engraving titled Melancholia, executed by the German artist Albrecht Dürer in 1514. The square itself is shown below.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Observe that this has inversion symmetry. A straight line segment connecting the centres of a pair of squares thus related passes through the centre of symmetry and is bisected by it. With two such pairs of squares, therefore, we get two line segments that bisect each other. Hence the centres of the four squares in question form the corners of a parallelogram. To obtain such a parallelogram we must choose two squares out of the eight in one half of the 4 x 4 grid. The matching squares (their 'mates') get selected automatically. Now we have 28 ways of choosing 2 objects out of 8. One can identify these 28 parallelograms (with centres at the centre of the grid) and thereby obtain 28 quartets of numbers giving the magic total. Four of these shapes are actually squares, while four others are non-square rectangles, two are non-square rhombuses, sixteen are general parallelograms, and two are straight lines (the diagonals) which can be considered collapsed or 'degenerate' parallelograms. Some of these parallelograms are displayed below.

	10	11	
	6	7	

16			13
4			1

	3		
			8
9			
		14	

16			
		11	
	6		
			1

The equality of row and column sums is not a consequence of the inversion symmetry. They are independently contrived by a judicious distribution of the numbers 1 to 8 in the grid. (The other numbers then get assigned automatically.)

Each row and each column shares a symmetry axis with the entire grid. Eight other quartets giving the magic sum and sharing a symmetry axis with the entire grid are as follows: the four squares in each quadrant of the main grid and the corners of four 3 x 3 squares.

There are sixteen other quartets giving the magic sum, and with at least one element of symmetry, but not placed symmetrically in the main grid. These are eight 2 x 3 rectangles, four 'kites' (2 erect and 2 inverted) and four arrowheads (2 erect and 2 inverted). The latter two patterns appear only in the vertical sense and have no horizontal counterparts.

There are 86 ways of obtaining a sum of 34 by choosing four numbers from 1-16. (It would be an interesting but challenging exercise for the student to verify this.) Durer's magic square exhibits 60 of these in symmetrical patterns. (The student is invited to verify this as well.)

An Indian magic square

As a counterpoint to the magic square discussed above we look at a magic square of Indian origin.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

It does not have the inversion property and so does not exhibit many properties that follow from it. However, apart from the row, column and diagonal property it has other interesting features.

Quartets of numbers forming the corners of eight isosceles trapeziums add to the magic sum. An example is given below.

	12	1	
16			5

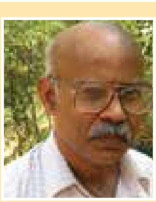
It has the 'pandiagonal' property, that is, quartets formed from numbers on the broken diagonals give the magic sum. An example follows.

		1	
	13		
16			
			4

Every 2 x 2 square gives the magic sum, as does the set of corners of each 2 x 4 rectangle.

The magic square can be extended by repetition in both East-West and North-South directions to give a 'Magic carpet' – an open 2-D array of numbers, where any four neighbouring numbers in a line (vertical, horizontal or diagonal) or forming a 2 x 2 square yield the magic sum. In addition, numbers at the corners of any 3 x 3 square and any 4 x 4 square yield the magic sum.

Further investigations in this area will surely prove to be a 'magic carpet ride' for a young mathematician or puzzle enthusiast.



A RAMACHANDRAN has had a long standing interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for over two decades, and now stays in Chennai. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.