

Covering the Plane with Repeated Patterns - Part I

“Hey mummy, look there. It is so beautiful!”

“Yes, my dear, it is indeed. It’s the epitome of Indian architecture. It has tessellations and many symmetrical patterns”

Architecture speaks of its time and place, but yearns for timelessness. Through this two part article, I will dwell on a topic that closely links mathematics with art and culture.

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Recently, my daughter and I happened to take a trip to the historical places of Delhi in the famous Hop-on-Hop-off bus. In our visit to various historical places we not only admired the marvels of the civil engineering of the past, we were also deeply impressed by the complexity and beauty with which our ancestors had blended geometry and art.

Among the interesting patterns that left us really in awe were the intricate designs we found on many pavements, ceilings and walls. We noticed that in most of these patterns a group of motifs were joined together such that they covered the entire plane. It was interesting to see how a single block of motif (s) fitted so well with each other without any gaps so as to extend in all the directions of the plane. We noticed several such designs in almost all the historical places that we had visited. I have attached some pictures that we took.

Keywords: Pattern, tessellation, quadrilateral, triangle, architecture, tile, Mughal, Escher

In this article I wish to share the mathematical ideas behind these perfectly fitting motifs. We will see how these complicated patterns can be broken into simpler components. This exercise can be

taken up with middle grade students to encourage them to integrate mathematics with culture and art. To start with, a quick look at these beautiful images:



A partition at Qutub Minar



Delhi Metro Station, Vishwavidayala



Ceiling at Red Fort



Partition at Qutub Minar



Net partition at VC office, DU



Wall at Qutub Minar



A house in Chandni Chowk



Carpet in Bangla Sahib Gurudwara



Jama Masjid



Net Partitions at Red Fort



A house in Chandni Chowk



Glimpse of Purani Dilli



Wall in Qutub Minar



Wall in Red Fort



Wall partition at Taj Mahal, Agra



Jahangir Mahal, Agra Fort

In all these pictures, you will notice that there is either one shape or a collection of shapes being repeated on a planar surface; the shapes fill the plane, with no gaps and no overlaps. The pattern can be extended in all directions of the plane. Such designs are known as **tilings or tessellations** (Greek: 'tessell' or 'tesera' which means 'tile'). The defining features of a tessellation are: (a) an infinite collection of congruent shapes, (b) the shapes fit together to fill a two-dimensional plane with no gaps and no overlaps.

The study of tessellations cuts across mathematics, art, architecture, culture and history. Tessellations can be found not only in man-made monuments but also the natural world – in the

honeycombs of bees, the skins of snakes and the shells of tortoises. Today, the topic has relevance in scattered fields such as x-ray crystallography, quantum mechanics, cryptology and minimization of waste material in the cutting of metal sheets.

One of the first mathematical studies of tessellations was conducted by Johannes Kepler in 1619, emanating perhaps from his study of snowflakes. More than two centuries later, in 1891, Russian crystallographer E. S. Fedorov gave the connection between isometries and tessellations. Maurits Cornelius Escher (1898-1972) was amongst the pioneers to look into tessellating patterns in detail; he created many masterpieces of his own.

Enumerating regular and semi-regular tessellations

Before proceeding to complicated tessellating patterns, we must study the tessellations of basic shapes – so-called **regular tessellations**. To fully understand the subject matter, you will need to experiment on your own with these shapes. For this, keep at least 10 copies each of the regular polygons with sides 3, 4, 5, 6, 8, 10 and 12. Remember to keep the same edge length for all the polygons!

Start with the smallest regular polygon: an equilateral triangle. Experimentally test if these shapes fit with each other leaving no gaps and no overlaps. You will find that they do fit with each other (Figure 2a). A square tessellates as well (Figure 2b). Regular pentagons, however, do not fit with each other; they leave gaps (Figure 3a). Regular hexagons do fit with each other (Figure 2c), and the pattern extends infinitely over the plane. Regular heptagons overlap (Figure 3b) and thus do not tessellate. You may now test if the regular nonagons (9-sided), regular decagons (10-sided) or regular dodecagons (12-sided) tessellate or not.

So what makes just three regular polygons tessellate? To ensure no gaps and no overlaps, the polygons must fit with each other to make an exact 360° around each vertex. Only in the case of equilateral triangles, squares and regular hexagons can this be done, as their interior angles (60° , 90° and 120° respectively) are divisors of 360° . In other regular polygons, the interior angles are not divisors of 360° , and the shapes do not tessellate. For example, in the case of a regular pentagon, the interior angle is 108° . When we place three such pentagons together, we find a gap of $360^\circ - 3 \times 108^\circ = 36^\circ$. For regular polygons with more than six sides we find an overlap, as the sum of three interior angles exceeds 360° . So these shapes will not tessellate.

The next question is: Can we find *combinations* of regular polygons which tessellate? Will a combination of squares and equilateral triangles tessellate? How about a combination of regular pentagons and regular heptagons? Can you predict which combinations of regular polygons will tile and which will not? Table 1 shows the interior angles of regular polygons.

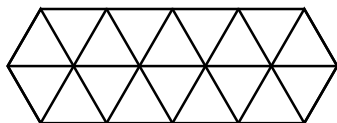


Figure 2a

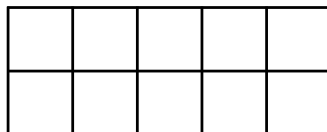


Figure 2b

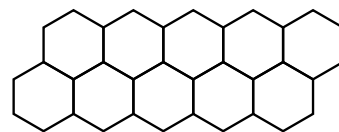


Figure 2c

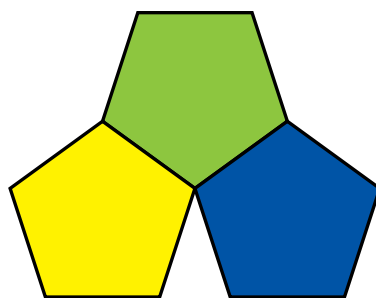


Figure 3a

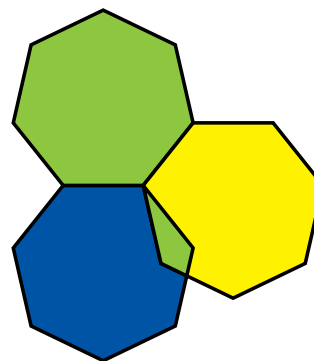


Figure 3b

Sides	3	4	5	6	8	9	10	12	18	20	n
Angle	60°	90°	108°	120°	135°	140°	144°	150°	160°	162°	$180^\circ(n-2)/n$

Table 1: Interior angles of various regular polygons

S. No.	1	2	3	4	5
Combination	(3.3.3.3.3.3)	(3.3.3.4.4)	(3.3.3.3.6)	(4.4.4.4)	(3.4.4.6)

S. No.	6	7	8	9	10	11
Combination	(3.3.6.6)	(3.3.4.12)	(6.6.6)	(5.5.10)	(4.8.8)	(4.6.12)

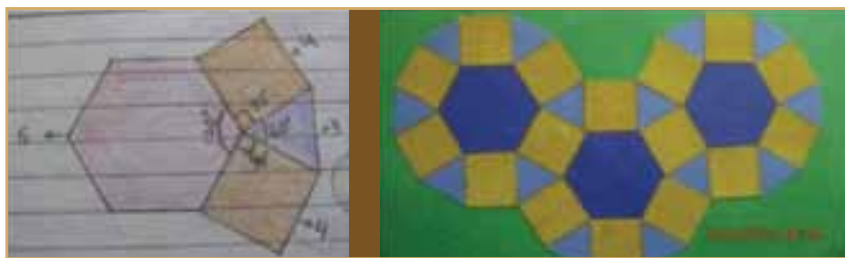
S. No.	12	13	14	15	16	17
Combination	(4.5.20)	(3.12.12)	(3.10.15)	(3.9.18)	(3.8.24)	(3.7.42)

Table 2: Combination of polygons that fill 360° at a vertex

By simply adding angles, we ought to be able to predict the combinations of polygons that *may tessellate*. Three equilateral triangles and two squares may tessellate, as their interior angles add up to 360° . So also an equilateral triangle, a regular decagon and a regular 15-sided polygon; two pentagons and a decagon may tessellate. By

analysing the possibilities, we find that there are 17 different combinations of regular polygons that *might* fit together at a vertex to make an angle of 360° . These possibilities are listed in Table 2.

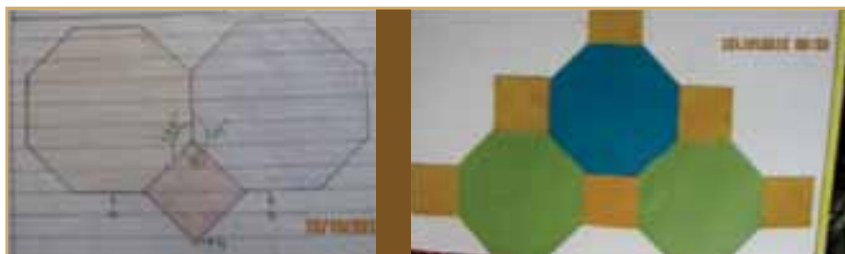
The notation in Table 2 is as follows. Consider the combination (3.7.42). This means that each vertex



(3.4.6.4) combination and its tessellation



(3.6.3.6) combination and its tessellation



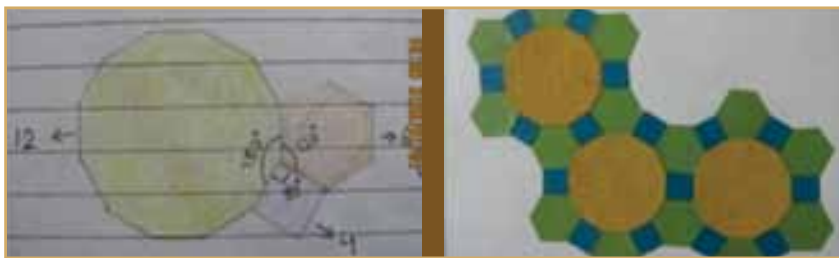
(4.8.8) combination and its tessellation

in the tessellation is shared by an equilateral triangle, a regular 7-sided polygon and a regular 42-sided polygon. Similarly, for the combination (3. 4. 4. 6) each vertex of the tessellation is shared by an equilateral triangle, two squares and a regular hexagon.

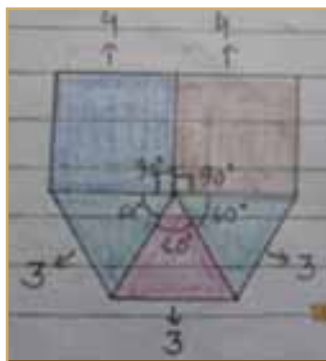
Listing the possibilities using arithmetical reasoning tells us only half of the story. Example: We have listed (3. 3. 3. 4. 4) as a possibility since $60^\circ + 60^\circ + 60^\circ + 90^\circ + 90^\circ = 360^\circ$. But this does not mean that we can actually exhibit a tessellation with this pattern. So we must ascertain *experimentally* if each of the patterns tessellates. Here are a few of the findings:

Some surprises are in store for us when we do the experimentation. For example, it may happen that a particular combination that yields 360° on summing gives rise to different spatial arrangements. Example: The interior angles of three equilateral triangles and two squares yield a complete angle ($3 \times 60^\circ + 2 \times 90^\circ = 360^\circ$), but this combination yields two possible spatial arrangements (Figure 4a and 4b).

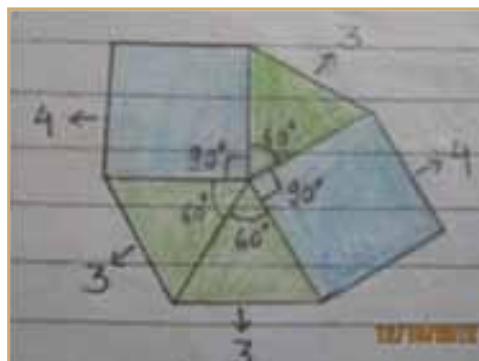
Will both arrangements yield valid tessellations, with the same spatial arrangement being maintained at all vertices? (Such a tessellation is called a **semi-regular tessellation**.)
Yes, they do!



(4. 6. 12) and its tessellation



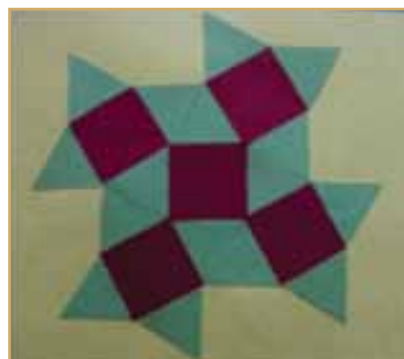
(3. 3. 3. 4. 4)
Figure 4a



(3. 4. 3. 3. 4)
Figure 4b



(3. 3. 3. 4. 4)



(3. 4. 3. 3. 4)

In other cases we find that the opposite happens!
Example: Consider the case (3.10.15). It is arithmetically feasible, but by experimentation we find that it does not tessellate.

(Editor's note. The mathematical note following this explains how to find all possible semi-regular tessellations, and why certain combinations like (3.10.15) do not tessellate.)

In Part-II of this article we show how to modify regular tessellations and produce attractive artwork.

Further Reading

- i. <http://library.thinkquest.org/16661/index2.html#anchor-top>
- ii. http://euler.slu.edu/escher/index.php/Tessellations_by_Recognizable_Figures
- iii. <http://www.mcescher.com> (biography and works of M.C. Escher)
- iv. <http://www.mcescher.com/Gallery/gallery-symmetry.htm>
(Escher's tessellations that can be taken up for class discussions)

Sources of some pictures

- i. <http://www.mcescher.com/Gallery/gallery-symmetry.htm>

Acknowledgements

The photographs have been taken by me and my B Ed students (batch 2012-13). I acknowledge my students' efforts in making intricate tessellations. I extend my thanks to them for allowing me to use their designs for this article.



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